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Models of Dark Energy

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We introduce models of dark energy.

I. INTRODUCTION

This is a short 3-hours lecture given at the ISAPP summer school on dark matter and dark energy in Sorrento, Italy. I tried to introduce several important classes of dark energy models and their theoretical implications. The references are surely far from complete and mostly serve as examples. Maybe the best way to get an overview over the subject is to read a review on dark energy, for instance [1–3].

A. Preliminaries

In order to discuss dark energy models, let us briefly summarize the equations governing the evolution of a Universe obeying Einstein's equation. In general relativity, Einstein's equations relate the geometry of the universe locally to the energy momentum content. The geometry is expressed via the metric $g_{\mu\nu}$ and subsequently through the Ricci Tensor $R_{\mu\nu}$ and the curvature scalar R, while the energy momentum tensor is commonly denoted by $T_{\mu\nu}$. Using the reduced Planck mass $\mathrm{M_P} \equiv (8\pi G)^{-1/2}$, Einstein's equations read[1]

$$\mathrm{M_P^2}\left(R^{\mu}{}_{\nu} - \frac{1}{2} g^{\mu}{}_{\nu} R\right) = T^{\mu}{}_{\nu}. \tag{1}$$

This famous equation can be obtained by varying the action

$$S = \int \mathrm{d}^4 x \sqrt{-g}\left[M_P^2 R + \mathcal{L}_m\right], \tag{2}$$

with respect to the metric $g_{\mu\nu}$, where the matter Lagrangian contains the standard model of particles plus dark matter etc,

$$\mathcal{L}_m = i\bar{\Psi}\nabla\!\!\!/\,\Psi + ie\bar{\Psi}A\!\!\!/_{\mu}\Psi + \ldots \tag{3}$$

In order to solve the very complicated, coupled differential Einstein equations analytically, one needs to guess the geometry of the space and hence the metric. The most general metric that is isotropic and homogenous on constant time hyper-surfaces is the Robertson-Walker metric. For spatially flat geometries (the case we will concentrate on), the metric can be written in terms of the coordinates x^i as

$$ds^2 = a(\tau)^2\left(-\mathrm{d}\tau^2 + \delta_{ij}\mathrm{d}x^i \mathrm{d}x^j\right), \tag{4}$$

where τ is the conformal time which is related by $\mathrm{d}\tau = \mathrm{d}t/a$ to the usual time. The expression 'conformal time' is well chosen, for the metric (4) is conformally related to the usual Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ by the conformal factor $a(\tau)$. For flat cosmologies, we normalize $a(\tau)$ such that today, we have $a_0 \equiv a(\tau_0) = 1$, where here and in the following a subscript 0 will denote quantities as measured today. As $a(\tau)$ determines the stretching of *physical* length scales,

$$l^2_{\mathrm{physical}}(\tau) = g_{ij} l^i l^j = a(\tau)^2 \delta_{ij} l^i l^j = a(\tau)^2 \boldsymbol{l}^2, \tag{5}$$

it is commonly called the *scale factor*. Please note that 3-vectors are in bold, spatial components are denoted by Latin indices and the 3-vector scalar product is the usual one: $\boldsymbol{x}\,\boldsymbol{y} = \delta_{ij} x^i x^j$. It is common practice to describe the matter content of the universe by fluids. Even in cases where this description is no longer valid and one needs to think in terms of distribution functions, we will still *identify* certain parts of these distributions with fluid terminology. For a start, let us briefly forget about cases where the fluid description breaks down and note that the energy momentum tensor for a *perfect* fluid is [14]

$$\bar{T}^{\mu}{}_{\nu} = \mathrm{diag}(-\bar{\rho}, \bar{p}, \bar{p}, \bar{p}), \tag{6}$$

where $\bar{\rho}(\tau)$ is the (unperturbed[2]) energy density and $\bar{p}(\tau)$ is the pressure. The relation between $\bar{\rho}$ and $\bar{p}$ is expressed in the equation of state

$$\bar{p}(\tau) = w(\tau)\bar{\rho}(\tau). \tag{7}$$

For non-relativistic matter, the pressure vanishes, whereas photons and massless neutrinos have $w = 1/3$. From the $0-0$ and $i-i$ part of Einstein's Equation (1), we we get the Friedmann equation

$$3\mathrm{M_P^2} H^2 = \rho(\tau). \tag{8}$$

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[1] When required, the cosmological constant will be assumed to be part of the energy momentum tensor.

[2] Anticipating perturbation theory, we denote background quantities by a bar.

Here, the Hubble parameter H is related to the scale factor $a(\tau)$ by

$$H \equiv a^{-1}\frac{\mathrm{d}a}{\mathrm{d}t} = a^{-1}\frac{\mathrm{d}a}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} \equiv a^{-2}\dot{a}, \tag{9}$$

where a dot denotes a derivative with respect to conformal time τ throughout this lecture.

The ratio of the energy of some species ρ to the so called critical energy density $\rho_{crit.} \equiv 3\mathrm{M}_\mathrm{P}^2 H^2$ is defined as

$$\Omega \equiv \frac{\rho}{\rho_{crit.}} = \frac{\rho}{3\mathrm{M}_\mathrm{P}^2 H^2} \tag{10}$$

For a flat universe, Ω is just the fraction a given species contributes to the total energy of the Universe. Careful: while Ω is a function of time, the subscript 0 indicating today's value is frequently omitted in the literature.

Conservation of the zero component of the energy momentum tensor,$\nabla_\mu \bar{T}^\mu_{\ 0} = 0$, yields the useful relation

$$\frac{\dot{\bar{\rho}}}{\bar{\rho}} = -3(1+w)\frac{\dot{a}}{a}. \tag{11}$$

The solution to the above energy conservation equation is rather simple for $w = const$:

$$\mathrm{d}\ln\bar{\rho} = -3(1+w)\mathrm{d}\ln a$$

upon integrating leads to

$$\ln\frac{\bar{\rho}}{\bar{\rho}_0} = -3(1+w)\ln\frac{a}{a_0},$$

and exponentiating

$$\frac{\bar{\rho}}{\bar{\rho}_0} = \left[\frac{a}{a_0}\right]^{-3(1+w)},$$

so for $a_0 = 1$ this is

$$\bar{\rho} = \bar{\rho}_0 a^{-3(1+w)}$$

Finally, by combining Friedmann's equation (8) with the $i-i$ part of Einstein's equation one obtains

$$\sum_{\text{all species}}\left(\bar{\rho}\left[\frac{1}{3}+w\right]\right) = -2\mathrm{M}_\mathrm{P}^2\, a^{-1}\frac{\mathrm{d}^2 a}{\mathrm{d}t^2}. \tag{12}$$

This last equation tells us that in order to have an accelerated expansion of the Universe, we would need at least

$$w < -\frac{1}{3}.$$

B. Observation: Something is missing

There are three main observations leading us to believe that some sort of dark energy is needed to explain our Universe in the Einstein framework:

- Sne Ia observations measure the luminosity distance d_l to some redshift z. They yield an accelerated Universe:

$$\frac{\mathrm{d}^2 a}{\mathrm{d}t^2} > 0$$

- Cosmic microwave background (CMB): Curvature negligible $\rightarrow$ $\Omega_{tot.} \approx 1$ at the same time, $\Omega_m \approx 0.3 \rightarrow \Omega_{rest} = 0.7$
- Structure formation: given CMB calibration, there is too much structure if $\Omega_m = 1$. One way out would be a substance that does not cluster like matter.

C. And now?

Given the observations, there are several routes to explore. It could be that ...

- We are living in an under-dense bubble within our universe
- The non-linearly clumped matter leads to gravitational forces that trick us into believing that the Universe accelerates, because we analyze it using a homogenous Universe plus small fluctuations. This is called back-reaction.
- The Einstein action is a more complicated function of R etc., e.g. $\mathcal{L} = M_P^2 R + \alpha R_{ab}R^{ab} + \beta R_{abcd}R^{abcd}$ or $\mathcal{L} = \mathrm{M}_\mathrm{P}^2 R + \mathrm{M}_\mathrm{P}^2\frac{\mu^4}{R}$.
- Even more modifications of gravity are needed: Modified Newtonian Dynamics (MOND)
- There is a missing component in the matter Lagrangian, called Dark Energy

In this lecture, we will mainly discuss the last topic, namely dark energy models. They are, however, related to modifications of the Einstein action which we will point out later.

II. DARK ENERGY MODELS

Maybe the simplest dark energy model is a so called cosmological constant, invented by Einstein.

A. The cosmological constant

Einstein invented the cosmological constant on the left hand side of his equation:

$$\mathrm{M}_{\mathrm{P}}^2\left(R^{\mu}{}_{\nu}-\frac{1}{2}g^{\mu}{}_{\nu}R-\Lambda g^{\mu}{}_{\nu}\right)=T^{\mu}{}_{\nu}. \tag{13}$$

However, it is nowadays brought to the right hand side and absorbed in $T^{\mu}{}_{\nu}$, because it is identified with the vacuum energy of quantum field theory. In terms of our fluid language, a cosmological constant is

$$\begin{aligned} T^{\mu}{}_{\nu} &= \Lambda\ \mathrm{diag}(1,1,1,1) && (14)\\ &= \mathrm{diag}(-\rho,p,p,p), && (15)\end{aligned}$$

so $w=p/\rho=-1$ for a cosmological constant. But what does Quantum Field Theory say? Consider some non-interacting quantum field Φ which you can consider as infinitely many harmonic oscillators. The relativistic energy-momentum relation is

$$\omega_k=\sqrt{\boldsymbol{k}^2+m^2}$$

and in order to get the vacuum contribution, we need to sum over all energies ω. In order to get a finite result, we have to stop at some ultra violet cut-off Λ. Let's do it!

$$\begin{aligned} \rho_{vac} &= \int_0^{\Lambda}\frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_k}\frac{1}{2}[\omega_k^2+\boldsymbol{k}^2+m^2] && (16)\\ &= \int_0^{\Lambda}\frac{\mathrm{d}^3k}{(2\pi)^3 2\omega_k}\omega_k^2 && (17)\\ &= \int_0^{\Lambda}\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2}\omega_k && (18)\\ &\approx \int_0^{\Lambda}\frac{\mathrm{d}^3k}{(2\pi)^3}\frac{1}{2}|\boldsymbol{k}| && (19)\\ &= \frac{4\pi}{(2\pi)^3}\int_0^{\Lambda}k^2\mathrm{d}k\,\frac{1}{2}k && (20)\\ &= \frac{1}{4\pi^2}\int_0^{\Lambda}\mathrm{d}k k^3 && (21)\\ &= \frac{1}{16\pi^2}\Lambda^4 && (22)\end{aligned}$$

Now, what is Λ^4? Surely, it must be some high-energy-physics scale. A very high scale at which we know some more fundamental theory has to take over is the Planck scale $\mathrm{M_P}\sim 10^{18}$Gev

$$\Lambda^4\approx \mathrm{M}_{\mathrm{P}}^4\approx 10^{120}H^2.$$

Our estimate for ρ_{vac} is 120 orders of magnitude larger than the energy density of our current Universe! Provided there is super symmetry before M_P, this incredible mismatch is smaller, but still very substantial. As you might know, bosonic and fermionic fields give opposite vacuum contributions. So if there is SUSY at 10^3Gev, the mismatch is "only" 60 orders of magnitude. Nevertheless, a cosmological constant matches observations very well. Yet, no one has offered a satisfactory answer, why the vacuum contribution of the quantum files should almost, but not entirely cancel. After all, we know that QFT works very well at least up to 10^3Gev and loop contributions are essential in the computation of cross sections etc. In addition, the question arises, why the cosmological constant should be relevant for the expansion of the Universe right at our redshift and why matter and dark energy are of comparable importance. These are the so called "Why now?" or "Coincidence" problems. In a nutshell, a cosmological constant ...

- matches observations well
- is simple
- currently nightmare for theoreticians
- does not solve coincidence and why-now problem

1. *The Landscape*

Did I say that no-one had explanation for Λ? Well, many string theorists would disagree. There is a plausible answer, called the string landscape. As you might have heard, there are many distinct ways to curl up the extra dimensions of string theory. In 2000, Raphael Bousso and Joe Polchinski [6] showed that there might be up to 10^{500} different string vacua. Among other things, each of those would have a different cosmological constant. In this picture, our small cosmological constant is just one out of the 10^{500} possibilities and pure chance. The antrophic principle is then applied to explain why we do not live in a universe with large cosmological constant. As a positive cosmological constant shows up as a repulsive force, a slightly larger Λ would prevent the formation of stars and galaxies, precluding human live.

B. Scalar field dark energy models

At least since inflation was invented, scalar fields play an important role in cosmology. A good reason is that they are particularly simple to work with – fermions are a nightmare in curved space compared to scalars. Another reason is that we seem to know one scalar field at least: the Higgs. Yet, while LHC will most probably find the Higgs, it could well be that there is no fundamental Higgs at all, but that the Higgs is a condensate of fermions

$$\Phi=\langle\bar{\Psi}\Psi\rangle.$$

The same is true for scalar dark energy models: they do not necessarily have to be fundamental scalar particles. The scalars could be condensates or anything else that can be described by a scalar field. Essentially, any quantity that in our frame evolves as a function of time in the

Universe is on the level of the Lagrangian a scalar field, because it has to be a scalar quantity. Otherwise, we would break general relativistic invariance. The action for the scalar field is

$$S = -\int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + V(\varphi)\right],$$

and in a homogenous universe, $\varphi = \bar{\varphi}(\tau)$,

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2}a^{-2}\dot{\bar{\varphi}}(\tau)^2 - V(\bar{\varphi})\right].$$

When you compute the energy momentum tensor from S, you will find that

$$\bar{\rho} = \frac{1}{2}a^{-2}\dot{\bar{\varphi}}(\tau)^2 + V(\bar{\varphi}) = T + V$$

and

$$\bar{p} = \frac{1}{2}a^{-2}\dot{\bar{\varphi}}(\tau)^2 - V(\bar{\varphi}) = T - V$$

where T and V are kinetic and potential energy of the field. The equation of state is

$$w = \frac{\bar{p}}{\bar{\rho}} = \frac{T+V}{T-V},$$

and we see that as $T \to 0$, $w \to -1$. So in the limit that the scalar field does not move, it resembles a cosmological constant. It is, however, much more versatile. It can for instance have $w = 1/3$ in the early Universe and $w = -0.95$ now. Suppose it had $w = 0$, which is the equation of state of matter. From energy conservation, we know that its energy dilutes like matter when the universe expands. Does this mean that a scalar field with $w = 0$ behaves like matter and forms clumps, stars, galaxies etc? The answer is no. Inspecting linear perturbations around its mean value $\bar{\varphi}$, one finds that perturbations on scales within the horizon have a speed of sound that is the same as that of light

$$\frac{\delta p}{\delta \rho} = 1 + rest,$$

where *rest* is zero in the rest-frame of the dark energy fluid and tends to zero on microscopic scales (microscopic meaning well inside the horizon!). This speed of sound tells us that you cannot localize a heap of dark energy in some spot. The overdensity would immediately free-stream with the speed of sound and dilute.

1. Attractors

For some potentials [19–21] the evolution of the scalar field is independent of its initial conditions. This is rather nice, because one does not have to fine-tune initial conditions. In particular, exponential potentials lead to an equation of state of dark energy that tracks that of the other species in the universe. During radiation domination, it adjusts to $w^\varphi = 1/3$, whereas during matter domination $w^\varphi = 0$. Hence, the value of Ω_d remains essentially the same from early times to today. This can in principle help to avoid the coincidence problem. Unfortunately, a simple exponential potential model does not lead to acceleration.

2. Working backwards

As we have seen, given a Lagrangian together with initial conditions (or an attractor solution), we can infer the equation of state, say as a function of redshift $w(z)$. But you can also go the other way around and reconstruct the scalar field Lagrangian, provided you know $w(z)$ and the dark energy abundance today Ω_0^φ. The trick is simple: First, reconstruct $\rho_{d.e.}(z)$ by working backward using energy conservation

$$\mathrm{d}\rho_{d.e.} = 3[1 + w(z)]\rho_{d.e.}\frac{\mathrm{d}z}{z+1}.$$

Then, express the potential and kinetic terms using w and $\rho_{d.e.}$:

$$T(z) = \frac{1+w(z)}{2}\rho_{d.e.}(z) \quad (23)$$

$$V(z) = \frac{1-w(z)}{2}\rho_{d.e.}(z) \quad (24)$$

From $T = a^{-2}\dot{\bar{\varphi}}^2/2$, we get

$$\dot{\bar{\varphi}} = \sqrt{2a^2T} \quad (25)$$

and integrating this, you obtain $\varphi(z)$ and hence $V(\varphi(z))$.

3. k-essence

Actually, one can extend the analysis to more general forms of scalar field Lagrangians without complicating the discussion too much [22]. The trick is to look at Lagrangians of the form

$$S = \int \mathrm{d}^4x\sqrt{-g}F(X,\varphi),$$

where $X \equiv \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi$. Please note that with our metric conventions, $X = -T$ for a homogenous field, i.e. X is the kinetic energy up to a sign. In terms of this Lagrangian, the equation of state is given by

$$w = \frac{F}{2XF_{,X} - F}$$

and the speed of sound is

$$c_s^2 = \frac{F_{,X}}{F_{,X} + 2XF_{,XX}}$$

The speed of sound connects the pressure perturbation to the energy perturbation via [23] (equation holds in Newtonian as well as synchronous gauge)

$$\delta p = c_s^2 \delta\rho + k^{-2}\bar{\rho}\, v\,(1+w)\left[3\frac{\dot{a}}{a}(c_s^2 - w) + \frac{\dot{w}}{1+w}\right]$$

Here, v is the bulk velocity of the dark energy fluid and k the Fourier mode, i.e length scale of the perturbation. Please note that the speed of sound is not the adiabatic speed of sound which is defined as $c^2_{s,adiab.} = \dot{p}/\dot{\rho}$.

As you can convince yourself, both expressions reduce to the ones for a canonical scalar field which has $F = X + V(\varphi)$.

Quite often, one looks at Lagrangians of a particular form

$$F = -\mathcal{L} = k(\varphi)X + V(\varphi).$$

Armenderiaz-Picon, Mukhanov and Steinhardt have shown that one can construct k-essence models that at least to some degree solve the why-now problem. In the early universe, the models are in an attractor solution tracking radiation. When matter takes over from radiation, the attractor vanishes and the field starts to accelerate the universe.

4. A Phantom Menace

As said, a canonical scalar field cannot give $w < -1$. Before 1999, there were actually few people thinking about such a possibility. There are many reasons why one shouldn't like $w < -1$: the energy density would *grow* when the universe expands. Nothing we know, not even vacuum energy does this. On top of that, this violates the null dominant energy condition, which is at the foundation of several theorems in general relativity. There are some more reasons, I'll mention them soon. Nevertheless, let us be humble and unprejudiced and observe what Robert Caldwell did in 1999 [24]: experimentally, we are quite close to $w = -1$. Are we maybe even below -1? As a toy model, he flipped the kinetic term in the Lagrangian

$$F = -\mathcal{L} = -X + V(\varphi)$$

as we have learned above

$$w = \frac{F}{2XF_{,X} - F} \tag{26}$$

$$= \frac{-X+V}{-2X-(X+V)} \tag{27}$$

$$= \frac{-X+V}{-X-V} \tag{28}$$

$$= \frac{T+V}{T-V} \tag{29}$$

For $T \to 0$, we again get $w \to -1$. But for $T < V$, we can reach arbitrarily negative values of w. As far as the dynamics of the scalar field is concerned, it runs *up* the potential. Naively, one should expect that this leads to instability. However, small perturbations in the field behave exactly like in the canonical case. They do not grow, but free-stream with again the speed of light

$$c_s^2 = \frac{F_{,X}}{F_{,X} + 2XF_{,XX}} \tag{30}$$

$$= \frac{F_{,X}}{F_{,X}} = 1 \tag{31}$$

From the point of view of quantum field theory, phantom dark energy looks rather peculiar. Usually, it costs a field energy to have gradients. With the negative kinetic term, this situation is reversed. So it's actually favorable to have an ultra-violet mode. So if the phantom field is coupled to other particles, say at the least gravitons. It can decay thereby producing gravitons [25]. The same is true for other particles, which can decay to *heavier* particles, when they produce phantom particles at the same time. You can imagine that this is quite a situation!

C. Quintom Cosmologies

Now that we have ordinary scalar fields $w \geq -1$ and phantom fields $w \leq -1$, you can add 1 and 1 together. Surely, we can have fields that do both, you might say. The answer is yes and no. Yes, because noone can stop you from parameterizing $w(a)$ say according to

$$w(a) = -0.7 - (1+a)$$

This model would have $w = -2.7$ today and $w = -0.7$ in the early universe. Such models are called 'Quintom' models [26]. You wouldn't be able to reconstruct a meaningful canonical scalar field Lagrangian, because canonical scalar fields cannot have $w < -1$. But with $k - essence$, this is indeed feasible. So you can find a Lagrangian and *one* scalar field trajectory that would yield the requested $w(a)$ behavior. Nevertheless, it won't work in the real world [23, 27–29]. A quick way to see why even the k-essence field won't be able to cross the cosmological constant boundary is to look at an example that is slightly too simplistic for the most general case, but we will cover that in a second, too. So imagine that you can neglect the Hubble expansion for a moment. At the moment of crossing, say at $z_\star$, all dark energy is in $V(\varphi_\star)$. Neglecting the Hubble expansion, however, energy is conserved. So the only trajectory that can cross is the one with $E = V(\varphi_\star)$. All other trajectories of φ will be reflected at $w = -1$. More formally, as promised, consider the k-essence action

$$S = \int \mathrm{d}^4x\sqrt{-g}F(X,\varphi),$$

at $w = -1$, we have

$$-1 = w = \frac{F}{2XF_{,X} - F}$$

leading to

$$XF_{,X} = 0.$$

So either all gradients vanish $X = 0$, or $F_{,X} = 0$. There is nothing that forces $X = 0$ at the crossing, so the only way to manufacture a crossing is via $F_{,X_\star} = 0$. Computing the "cost" of having gradients at the crossing gives

$$\delta S = \int \mathrm{d}^4x\sqrt{-g}\, F_{,X\,|\,X=X_\star}\delta X = 0.$$

This tells us that we can have arbitrary gradients, i.e. arbitrarily different trajectories from point to point at the crossing. At least in the field language, this is the breakdown of perturbation theory. As such, this might not be a no-go. You could argue that there might be non-linear effects etc. This is still an open issue. But the divergences certainly prevent the consistent use of ordinary perturbation theory, which is the workhorse in cosmology.

D. Coupled Dark Energy

Given that matter and dark energy are roughly equally important today, it's natural to search for mechanisms to explain this. One such mechanism is achieved through coupling dark energy to matter. Now, coupling dark energy to ordinary baryonic matter is restricted because we would have seen an additional attractive scalar force[3]. Yet, a coupling to dark matter might be better suited. There is still an observational limit, because a coupling to dark matter would result in an attractive force between dark matter particles. Essentially such forces are restricted because too much would yield more cuspy profiles in galaxies. Nevertheless, couplings of roughly gravitational strength are allowed. So consider the Lagrangian [30]

$$-\mathcal{L} = X + V(\varphi) + im(\varphi)\bar{\Psi}\Psi + \dots$$

which contains a coupling between some Fermion Ψ and dark energy φ due to some functional dependence of the mass $m(\varphi)$. If that looks strange, it might help to take a look at the more familiar situation of a constant mass plus a Yukawa coupling

$$\begin{aligned} -\mathcal{L} &= X + V(\varphi) + im\bar{\Psi}\Psi + ih\varphi\bar{\Psi}\Psi + \dots && (32)\\ &= X + V(\varphi) + i[m + h\varphi]\bar{\Psi}\Psi && (33)\\ &= X + V(\varphi) + im(\varphi)\bar{\Psi}\Psi && (34)\end{aligned}$$

For the background a coupling leads to the following evolution equations for the energy density in matter and the dark energy field [3]:

$$\begin{aligned} \mathrm{d}\rho_{d.e.} + 3(1+w)\rho_{d.e.} &= \rho_m\left(\frac{1}{\rho_m\sqrt{-g}}\frac{\delta S_m}{\delta\varphi}\right)\mathrm{d}\varphi && (35)\\ \mathrm{d}\rho_m + 3(1+w_m)\rho_m &= \rho_m\left(\frac{-1}{\rho_m\sqrt{-g}}\frac{\delta S_m}{\delta\varphi}\right)\mathrm{d}\varphi && (36)\end{aligned}$$

where S_m is the matter action. What would be desirable are *scaling* solutions in which $\rho_{d.e.}/\rho_m$ remains constant. In contrast to uncoupled dark energy, though, there are scaling solutions that yield an accelerated expansion of the Universe on top of the scaling. Luca Amendola, Miguel Quartin,Shinji Tsujikawa and Ioav Waga [31] have shown that you can always re-define φ such that the coupling term is constant, i.e. you can achieve

$$1 = \frac{1}{\rho_m\sqrt{-g}}\frac{\delta S_m}{\delta\varphi}.$$

On top of that, they showed that the field Lagrangian for scaling solutions can always be written as

$$-\mathcal{L} = F(X,\varphi) = Xg(Xe^{\lambda\varphi}).$$

As you can see, the standard exponential potential dark energy is a special case of this, using $g = 1 + M_P^4/(Xe^{\lambda X})$,

$$\begin{aligned} F(X,\varphi) &= X\left[1 + \frac{\mathrm{M}_\mathrm{P}^4}{Xe^{\lambda\varphi}}\right] && (37)\\ &= X + \mathrm{M}_\mathrm{P}^4 e^{-\lambda X} && (38)\end{aligned}$$

Now, the best of all worlds and the solution to our dark energy coincidence problem that fits observations would be two scaling regimes. The first one with no acceleration yielding the usual matter regime. Being a scaling solution, we would not need to fine tune initial conditions. The second one with acceleration today. The first scaling solution should be a saddle point, i.e. meta stable that later leads to a transition to the final attractor which is accelerating. Unfortunately, they showed that it is impossible to have both scaling regimes in sequence with the setup as above. That does not mean that coupled dark energy is impossible. The late time evolution could, for instance be not an attractor, but the naively most beautiful solution seems impossible.

1. *quantum loops*

Coupling dark matter to dark energy faces another potential problem from quantum corrections. Let me mention that the 1-loop correction to the dark energy potential is given by [32]

$$V_{\text{1-loop}}(\varphi) = V(\varphi) + \frac{\Lambda^2}{32\pi^2}V''(\varphi) - \frac{\Lambda_{\text{ferm}}^2}{8\pi^2}\left[m_\mathrm{f}(\varphi)\right]^2. \quad (39)$$

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[3] Field theory tells us that the exchange of spin 0 fields is always attractive. The exception to the rule are phantom fields.

Here, Λ is the cut-off for dark energy loops, wheres Λ_{ferm} is the cut-off for fermion loops of fermions that couple to dark energy. Now, V is of the order of $M_P^2 H_0^2$ today, because it dominates the Friedman equation. Please keep in mind that this combination is quite tiny in particle physics unit. In the tracking regime, $V'' \sim H^2$ holds[4]. So even for $\Lambda = \mathrm{M_P}$, the quantum corrections from the scalar self coupling are at most of the order of V. In most cases, they can even be absorbed by a redefinition of the potential. For instance in the case of exponential potentials. Try it!

Much more troublesome are fermion loops: The mass of the fermion has high-energy physics scale. The same is true for Λ_{ferm}. So

$$V \sim M_P^2 H_0^2 \ll m(\varphi)^2 \Lambda_{\mathrm{ferm}}^2$$

and you can see that the quantum loop correction of the coupling term completely dominates the potential term. In other words: the coupling $m(\varphi)$ *determines* the potential. It's rather unlikely that we guess the low energy effective potential *and* coupling correctly when we write down a Lagrangian. The exception, by the way is again an exponential potential together with an exponential coupling. In this case $m^2(\varphi)$ can be $\propto V$:

$$m(\varphi) = \exp(-\lambda/2\varphi)$$

leads to

$$m^2(\varphi) = \exp(-\lambda\varphi) \propto V = \mathrm{M_P^4} \exp(-\lambda\varphi)$$

for an exponential coupling together with the correct exponential potential.

III. GENERALIZED CHAPLYGIN GASES

A few years ago, a phenomenological model to explain both dark matter and dark energy was quite popular. In the so called Chaplygin gas model, dark matter and dark energy are unified with the equation of state

$$p = -A\rho^{-\alpha},$$

where A and α are constants. Inserting this relation in the energy conservation equation, you'll find that

$$\rho(t) = \left[A + \frac{B}{a^{3(1+\alpha)}}\right]^{\frac{1}{1+\alpha}},$$

where B is an integration constant. As you can see, ρ will never become smaller than $A^{1/(1+\alpha)}$. Defining

$$\Omega_m^\star \equiv \frac{B}{A+B}$$

and

$$\rho_\star \equiv (A+B)^{\frac{1}{1+\alpha}},$$

this can be written as

$$\rho(a) = \rho_\star \left[(1-\Omega_m^\star) + \Omega_m^\star a^{-3(1+\alpha)}\right]^{\frac{1}{1+\alpha}}$$

For comparison, the usual dark matter plus dark energy fluid scales as

$$\rho(a) = \rho_\star \left[(1-\Omega_m) a^{-3(1+w)} + \Omega_m a^{-3}\right]$$

As you can see, for $\alpha = 0$, the GCG fluid becomes equivalent to Λ-CDM. So far, everything looks fine. The trouble, however, comes from linear perturbations. The speed of sound is

$$c_s^2 = \frac{\partial p}{\partial \rho} = -\alpha w.$$

While this looks good, because at early times with $w \approx 0$, we also have $c_s^2 \approx 0$ as needed for structure growth, it also means that unified dark matter/dark energy can have acoustic oscillations if $c_s^2 > 0$. Indeed, H.B. Sandvik et. al. [18] have shown that on scales

$$k \gtrsim \lambda_c^{-1} = \frac{aH}{\sqrt{-\alpha w}},$$

the density in a halo would be pressure supported and oscillate instead of grow. So if $w \to -1$ this means that we get a rough limit today on

$$|\alpha| \lesssim \left(\frac{H}{k}\right)^2,$$

where we could use a galaxy clustering scale of $k \sim 10h^{-1}Mpc$ which yields

$$|\alpha| \lesssim \left(\frac{10Mpc}{3000Mpc}\right)^2 \approx 10^{-5}.$$

So α needs to be very close to 0. In other words: the Chaplygin gas model behaves quite like ΛCDM.

IV. MODIFIED EINSTEIN ACTIONS AND SCALAR FIELDS

There is a connection between modifications to the Einstein action and scalar fields. I do not know who invented it, but since some of the transformations are called "Weyl"-scalings, I suppose that they must be almost as old as General Relativity. There is a useful account of what you can do with these transformations in an article by Wands [33], but there are others, too. The modifications I will present are quite limited for pedagogical reasons. If you read the article by Wands, you will

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[4] There are several ways to see this, the most intuitive one: in the scaling regime, the only available light mass scale that evolves with the expansion of the universe is the Hubble parameter.

find that you can accommodate much more complicated actions. So consider

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\mathrm{M}_\mathrm{P}^2 F(R) + \mathcal{L}_m\right]$$

with some function $F(R)$. We can introduce a scalar field coupled to R to get the following *scalar-tensor* action

$$S_{ST} = \int \mathrm{d}^4x\sqrt{-g}\mathrm{M}_\mathrm{P}^2\left[F(\varphi) + \frac{\partial F}{\partial\varphi}(R-\varphi)\right] + \mathcal{L}_m$$

Let us make a variation of S_{ST} with respect to φ!

$$\frac{\delta S}{\delta\varphi} \to \frac{\partial F}{\partial\varphi} + \frac{\partial^2 F}{\partial\varphi^2}(R-\varphi) - \frac{\partial F}{\partial\varphi} = 0$$

leading to

$$\frac{\partial^2 F}{\partial\varphi^2}(R-\varphi) = 0,$$

and for $F'' \neq 0$, this means that

$$R = \varphi.$$

So in classical theory, $R = \varphi$ and if you plug this back into S_{ST}, you recover our original action S. So they are both equivalent (at least on the classical level). Now make a mental note of this and consider the second part of our excursion: Suppose you had an action which you can (by suitable variable transformations) bring into a form that is called Jordan frame (another indication that this must be almost as old as GR)

$$S_J = \int \mathrm{d}^4x\sqrt{-g}\left[\Phi R - \frac{\omega}{\Phi}\Phi_{,\mu}\Phi_{,\nu} - \Lambda(\Phi) + \mathcal{L}_m\right]$$

Then make a conformal transformation

$$\tilde{g}_{\mu\nu} = \frac{\Phi}{\mathrm{M}_\mathrm{P}^2}g_{\mu\nu}.$$

Either you trust the literature, our you go ahead and compute what becomes of R and $\sqrt{-g}$ when you do this. The result is

$$S_E = \int \mathrm{d}^4x\sqrt{-\tilde{g}}\left[\mathrm{M}_\mathrm{P}^2\tilde{R} - \frac{1}{2}\tilde{g}^{\mu\nu}\psi_{,\mu}\psi_{,\nu} - V(\psi) + \frac{\mathrm{M}_\mathrm{P}^4}{\Phi^2}\mathcal{L}_m\right]$$

where we have defined a new field ψ

$$\mathrm{d}\psi = \sqrt{3+2\omega}\,\mathrm{M}_\mathrm{P}\mathrm{d}\ln\Phi,$$

i.e.

$$\frac{\Phi}{\mathrm{M}_\mathrm{P}^2} = \exp\left(\frac{\psi}{\mathrm{M}_\mathrm{P}\sqrt{3+2\omega}}\right)$$

and a new potential

$$V(\psi) = \Lambda(\Phi)\frac{\mathrm{M}_\mathrm{P}^4}{\Phi^2}$$

The action S_E is said to be in the Einstein frame, because gravitationally, this is the Einstein-Hilbert action. Let us add $1+1$ together and apply this to a model by Carroll, Duvvuri, Trodden and Turner [34]. Their action was

$$S = \int \mathrm{d}^4x\sqrt{-g}\,\mathrm{M}_\mathrm{P}^2\left[R - \frac{\mu^4}{R}\right] + \mathcal{L}_m$$

So the function is

$$F(\varphi) = \varphi - \frac{\mu^4}{\varphi}$$

and the derivative

$$F' = 1 + \frac{\mu^4}{\varphi^2}.$$

So according to our recipe, the equivalent scalar tensor action is

$$\begin{aligned} S_{ST} &= \int \mathrm{d}^4x\sqrt{-g}\mathrm{M}_\mathrm{P}^2\left[\varphi - \frac{\mu^4}{\varphi} + \left(1+\frac{\mu^4}{\varphi^2}\right)\{R-\varphi\}\right] + \mathcal{L}_m \\ &= \int \mathrm{d}^4x\sqrt{-g}\mathrm{M}_\mathrm{P}^2\left[R\left(1+\frac{\mu^4}{\varphi^2}\right) - 2\frac{\mu^4}{\varphi}\right] + \mathcal{L}_m \end{aligned} \quad (40)$$

identifying

$$\Phi \equiv \mathrm{M}_\mathrm{P}^2\left(1+\frac{\mu^4}{\varphi^2}\right),$$

this of the required Jordan form

$$S_J = \int \mathrm{d}^4x\sqrt{-g}\,\Phi R - 2\frac{\mu^4}{\varphi}\mathrm{M}_\mathrm{P}^2 + \mathcal{L}_m$$

which we can further 'standardize' by noting that

$$\sqrt{\frac{\Phi}{\mathrm{M}_\mathrm{P}^2} - 1} = \frac{\mu^2}{\varphi},$$

so

$$S_J = \int \mathrm{d}^4x\sqrt{-g}\,\Phi R - 2\mu^2\sqrt{\frac{\Phi}{\mathrm{M}_\mathrm{P}^2} - 1} + \mathcal{L}_m$$

According to our recipe, this is equivalent to the above Einstein action with the new field

$$\psi = \sqrt{3}\mathrm{M}_\mathrm{P}\ln\frac{\Phi}{\mathrm{M}_\mathrm{P}^2}$$

and a potential

$$\begin{aligned} V(\psi) &= 2\mathrm{M}_\mathrm{P}^2\mu^2\frac{\sqrt{\frac{\Phi}{\mathrm{M}_\mathrm{P}^2} - 1}}{\left(\Phi/\mathrm{M}_\mathrm{P}^2\right)^2} && (41) \\ &= 2\mathrm{M}_\mathrm{P}^2\mu^2\exp\left(-\frac{2}{\sqrt{3}}\frac{\psi}{\mathrm{M}_\mathrm{P}}\right)\sqrt{\exp\left(\frac{\psi}{\sqrt{3}\mathrm{M}_\mathrm{P}}\right)} && (42) \end{aligned}$$

Together with a coupling to *all* matter with strength

$$\exp\left(-\frac{2}{\sqrt{3}}\frac{\psi}{\mathrm{M}_\mathrm{P}}\right)\mathcal{L}_m$$

You should take away a few things from this:

- Very often, when people modify the Einstein action, they are essentially looking at a scalar quintessence model with coupling to matter
- What seems strange in one frame might be 'natural' in another
- Exponential potentials (and couplings) occur very naturally when you make such transformations

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