Quantum Mechanics
and
Neutrino Oscillations

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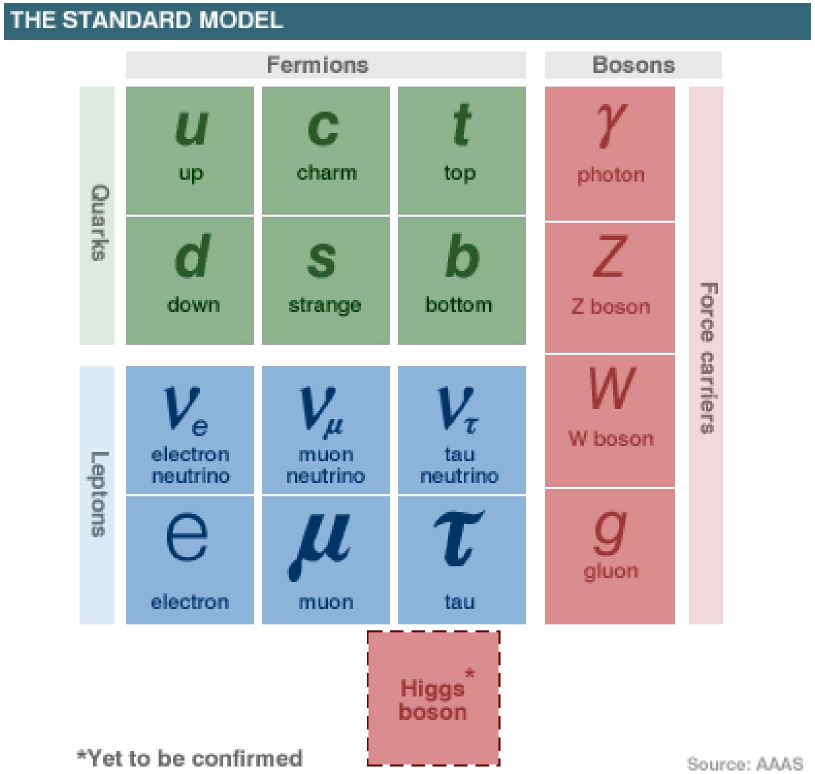
Program

1) Neutrino oscillations in vacuum
2) Neutrino oscillations in matter
    - Refractive index
    - Effective Hamiltonian
    - Constant density
    - Slowly varying density
    - Adiabatic propagation
    - Non-adiabatic evolution
    - Three generations

Neutrino flavors and interactions



THE STANDARD MODEL

	Fermions			Bosons	
Quarks	u up	c charm	t top	γ photon	Force carriers
	d down	s strange	b bottom	Z Z boson	
Leptons	ν_e electron neutrino	ν_μ muon neutrino	ν_τ tau neutrino	W W boson	
	e electron	μ muon	τ tau	g gluon	

Higgs boson*

*Yet to be confirmed

Source: AAAS

Flavors are related to defined interactions i.e., creation, absorption or scattering

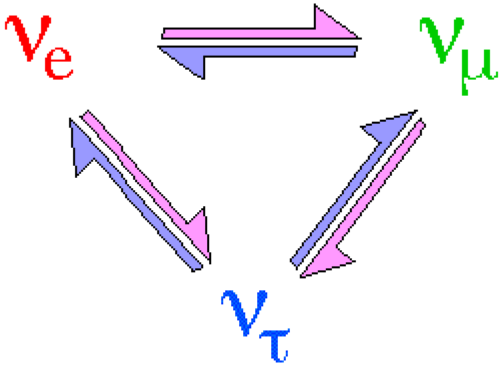
Examples:

$$n \longrightarrow e p \bar{\nu}_e$$

$$e^+ e^- \longrightarrow \nu_e \bar{\nu}_e$$

Neutrino Oscillations

Can neutrino flavors be transformed from one to another?
B. Pontecorvo

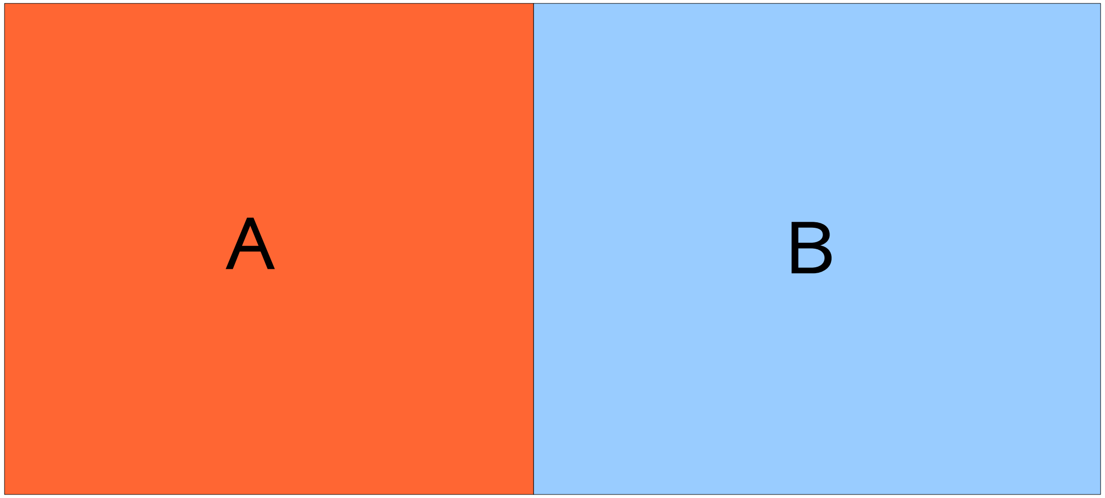


Need from some extra ingredients

mass and mixing

Simple analogy from Quantum Mechanics (two flavors)

Consider a particle that can be **detected** in one of the two sides of a box (A or B). These two states correspond to interactions. The corresponding quantum states are labeled |A> and |B>



Now suppose that the particle has some interaction that allows for transitions between A and B.

The Hamiltonian will contain an off-diagonal term to allow for A-B transitions. Of course, we can add diagonal terms later.

$$H = \begin{pmatrix} 0 & \Delta \\ \Delta & 0 \end{pmatrix}$$

Eigenvectors are not |A> and |B>.
Instead

$$|e_1\rangle = \frac{1}{\sqrt{2}}\left(|A\rangle - |B\rangle\right) \qquad |e_٢\rangle = \frac{١}{\sqrt{٢}}\left(|A\rangle + |B\rangle\right)$$

With eigenvalues $\mp\Delta$ respectively

In other words

$$\begin{pmatrix} |A\rangle \\ |B\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} |e_1\rangle \\ |e_2\rangle \end{pmatrix}$$

Assume the initial condition

$$|\Psi(0)\rangle = |A\rangle = \frac{1}{\sqrt{2}}(|e_1\rangle + |e_2\rangle)$$

$$|\Psi(t)\rangle = \frac{1}{\sqrt{2}}(\exp(\frac{-it\Delta}{\hbar})|e_1\rangle + \exp(\frac{it\Delta}{\hbar})|e_2\rangle)$$

Transition probability $P(A \to B; t) = \sin^2(\frac{t\Delta}{\hbar})$ oscillations!

Including diagonal terms in the Hamiltonian

$$\begin{pmatrix} |A\rangle \\ |B\rangle \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} |e_1\rangle \\ |e_2\rangle \end{pmatrix}$$

can be chosen
as real

$$|\alpha> = \sum_i U_{\alpha i} |e_i>$$

$$|e_i> = \sum_i U^\dagger_{i\alpha'} |\alpha'>$$

If the initial state is

$$|\Psi(0)>=|\alpha>$$

Then

$$|\Psi(t)>=\sum_{i,\alpha'}U_{\alpha i}e^{\frac{-itE_i}{\hbar}}U^{\dagger}_{i\alpha'}|\alpha'>$$

Giving a transition probability

$$P(\alpha\rightarrow\alpha';t)=|\sum_{i,\alpha'}U_{\alpha i}e^{\frac{-itE_i}{\hbar}}U^{\dagger}_{i\alpha'}|^2$$

Neutrinos : Let us substitute

$$|A>\rightarrow|\nu_e> \qquad |B>\rightarrow|\nu_\mu> \qquad \alpha = e,\mu$$

$$|e_i>\rightarrow|\nu_i> \qquad i = 1,2$$

$$\Delta \rightarrow (E_2 - E_1)/2$$

$$E = \sqrt{(pc)^2 + (mc^2)^2} \simeq pc + (mc^2)^2/(2pc)$$

Then

$$P(\nu_e \rightarrow \nu_\mu;t) = \sin^2 2\theta \sin^2 \frac{\pi t}{T_{osc}} \qquad T_{osc} = \frac{4\pi\hbar p}{(m_2^2 - m_1^2)c^3}$$

Neutrinos in matter

$$\hbar = c = 1$$

Particles propagating in a medium with a number density n acquire a refractive index related to the forward scattering amplitude

See e.g. Raffelt's book

$$E = \hbar\omega ; p = \hbar k$$

$$n_{ref} = 1 + \frac{2\pi}{\omega^2} n f_0(\omega)$$

real and imaginary part !

Imaginary part : total cross section (dispersion)
real part : forward propagation (coherent)

A different approach (closer to QFT) :
dispersion relations

$$(\omega - V)^2 = k^2 + m^2$$

Standard model effective interaction

$$H_{int} = \frac{G_F}{\sqrt{2}}[\bar{\Psi}_f \gamma_\mu (C_V - C_A \gamma_5) \Psi_f][\bar{\Psi}_\nu \gamma^\mu (1 - \gamma_5) \Psi_\nu]$$

$$n_{ref} = 1 \mp 2 C_V G_F \frac{n_f - n_{\bar{f}}}{\sqrt{2}\omega}$$

Example : electron neutrinos via
charged currents

$$V = \sqrt{2} G_F n_e$$

Two generations

$|\nu_e >$ and ”$|\nu_\mu >$”

Effective Hamiltonian

V is diagonal in flavor space

Kinetic energies are diagonal in mass eigenstates basis

$$H = U \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix} U^\dagger + \begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix}$$

$$E_i \simeq p + m_i^2/2p$$

Time evolution:
Schrödinger equation

$$i\frac{d}{dt}|\Psi >= H|\Psi >$$

Density will vary in space. For relativistic neutrinos $x \simeq t$

$$H = \begin{pmatrix} V - \frac{\Delta m^2}{4p}\cos 2\theta & \frac{\Delta m^2}{4p}\sin 2\theta \\ \frac{\Delta m^2}{4p}\sin 2\theta & \frac{\Delta m^2}{4p}\cos 2\theta \end{pmatrix}$$

(plus a diagonal matrix)

Constant density: V(x) = V

$$|\Psi> = \begin{pmatrix} a_e \\ a_\mu \end{pmatrix} = U_m \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta_m & \sin\theta_m \\ -\sin\theta_m & \cos\theta_m \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

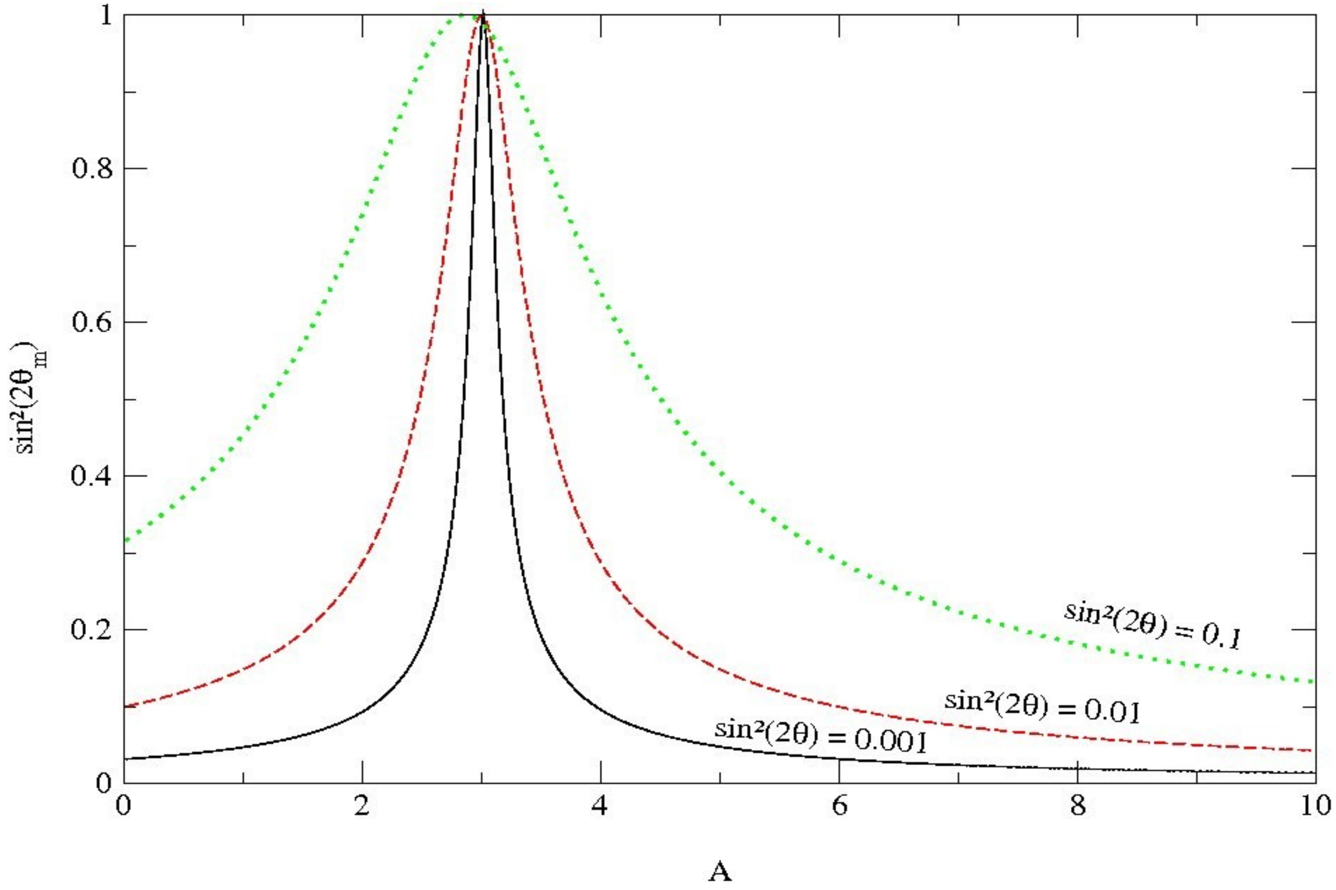
Matter mixing angle

$$\sin 2\theta_m = \frac{\Delta m^2 \sin 2\theta}{\sqrt{(2pV - \Delta m^2 \cos 2\theta)^2 + (\Delta m^2 \sin 2\theta)^2}}$$

Limiting cases

If V → 0 $\theta_m \to \theta$

$$A = 2pV \gg \Delta m^2 \cos 2\theta \qquad \sin 2\theta_m \to 0$$



Resonant behavior

Resonance position

$$A = \Delta m^2 \cos 2\theta$$

width

$$\Delta A = 2\Delta m^2 \sin 2\theta$$

Transition probabilities look like in vacuum, with the replacements

$$\theta \longrightarrow \theta_m$$

$$T_{osc} \longrightarrow L_m/c$$

$$P(\nu_e \to \nu_\mu; L) = \sin^2 2\theta_m \sin^2 \frac{\pi L}{L_m}$$

$$L_m = \frac{4\pi p}{m_2^2 - m_1^2}$$

Oscillation length in matter and mass eigenvalues

$$m_{1,2}^2 = \frac{1}{2}A \mp \frac{1}{2}\sqrt{(A - \Delta m^2 \cos 2\theta)^2 + (\Delta m^2 \sin 2\theta)^2}$$

Slowly varying density

For constant density, mass eigenstates evolve independently.
However, for a general density profile this is not true.

Let's go back to the Schrödinger equation, and
transform to local eigenstates:

$$\begin{pmatrix} a_e \\ a_\mu \end{pmatrix} = U_m(x) \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

diagonal

$$i \frac{d}{dx} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \left[U_m^\dagger H U_m - i U_m^\dagger \frac{dU_m}{dx} \right] \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

off-diagonal

In other words

$$i\frac{d}{dx}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -\frac{\pi}{L_m} & -i\frac{d\theta_m}{dx} \\ i\frac{d\theta_m}{dx} & \frac{\pi}{L_m} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\frac{d\theta_m}{dx} = \frac{1}{2}\frac{\Delta m^2 \sin 2\theta}{\sqrt{(A-\Delta m^2\cos 2\theta)^2 + (\Delta m^2 \sin 2\theta)^2}}\frac{dA}{dx}$$

Let us assume that $$|\frac{d\theta_m}{dx}| \ll \frac{\pi}{L_m} \qquad \forall x$$

adiabatic condition

This will be true if valid at the resonance, i.e. if

$$\gamma = \frac{\pi}{|L_m \frac{d\theta_m}{dx}|_{res}} \gg 1$$

Adiabaticity parameter

Under the adiabatic condition, matter eigenstates evolve independently (no transitions occur). However, the composition in terms of flavors (the mixing angle) will change with density.

Adiabatic propagation

Assume an electron neutrino is produced at $x = x_i$

and detected at $x = x_f$

$$|\nu(x_f) >= \sum_{\alpha',i} U_{\alpha',i}(\theta_i) e^{-i\int_{x_i}^{x_f} E_i(x)dx} U^*_{ei}(\theta_f)|\nu_{\alpha'}(x_i) >$$

(no crossing)

The final probability is

$$P(\nu_e \to \nu_\mu; x_f) = \frac{1}{2}(1 - \cos 2\theta_i \cos 2\theta_f - \sin 2\theta_i \sin 2\theta_f \cos \Phi)$$

with

$$\Phi = \int_{x_i}^{x_f} (m_2^2 - m_1^2) dx$$

For most situations, the last term can be averaged out, giving

$$\bar{P}(\nu_e \to \nu_\mu; x_f) = \sum_i |U_{\mu i}(x_f)|^2 |U_{ei}(x_i)|^2$$

$$= \frac{1}{2}(1 - \cos 2\theta_i \cos 2\theta_f)$$

Consider now adiabatic evolution
with the initial condition

$$A(x_i) \gg \Delta m^2 \cos 2\theta$$

Then

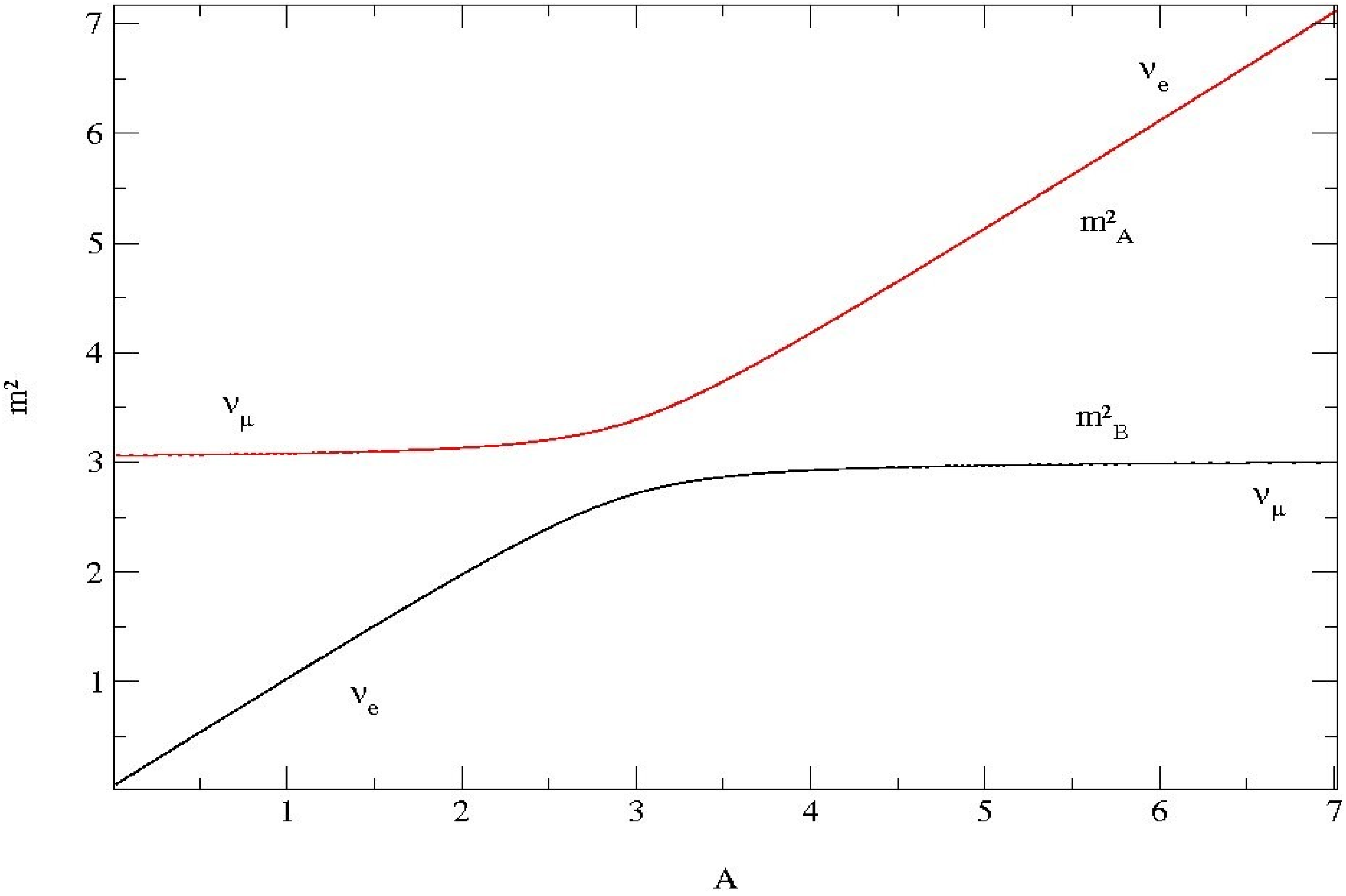
$$|\nu_e(x_i) > \simeq |\nu_2(x_i) > \qquad |\nu_\mu(x_i) > \simeq -|\nu_1(x_i) >$$

At low densities, if θ Is small

$$|\nu_2(x_f) > \simeq |\nu_\mu(x_f) > \qquad |\nu_1(x_f) > \simeq |\nu_e(x_f) >$$

MSW effect

(Mikheev, Smirnov and Wolfenstein)



Non-adiabatic evolution

Effect of level transitions (Landau-Zener)

1 ←→ 2

(crossing probability)

$$\bar{P}(\nu_e \to \nu_e; x_f) = \sum_{i,k} |U_{ei}(x_f)|^2 P_{ik} |U_{ek}(x_i)|^2$$

$$= \frac{1}{2} + (\frac{1}{2} - P_c) \cos 2\theta_i \cos 2\theta_f$$

$$P_{12} = P_{21} = P_c \qquad P_{11} = P_{22} = 1 - P_c$$

Calculation of the crossing probability

Only a few cases can be solved analytically:
- Linear profile
- Exponential profile
- Hyperbolic tangent

By expanding the density around the resonance, we obtain a linear approximation

$$A \simeq A_{res} - 2apx$$

(relative to resonance)

$$H = \begin{pmatrix} V - \frac{\Delta m^2}{4p}\cos 2\theta & \frac{\Delta m^2}{4p}\sin 2\theta \\ \frac{\Delta m^2}{4p}\sin 2\theta & \frac{\Delta m^2}{4p}\cos 2\theta \end{pmatrix}$$

Adding diagonal terms, one has

$$H = \begin{pmatrix} -ax/2 & c \\ c & ax/2 \end{pmatrix} \qquad c = \frac{\Delta m^2}{4p} \sin 2\theta$$

Two-level problem

- Quantum optics
- Molecular transitions
- Quantum information
- ...

$$H = -\frac{1}{2} ax\sigma_z + c\sigma_x \qquad [\sigma_z , \sigma_x] \neq 0$$

Analytical solution

With the substitutions

$$a_e(x) = W(z) \qquad z = \sqrt{a}e^{-i\pi/4}x$$

$$a_\mu(x) = g(x)e^{-\frac{i}{4}at^2}$$

One arrives to the equation

$$\frac{d^2W}{dz^2} + [\eta + \frac{1}{2} - \frac{z^2}{4}]W(z) = 0 \qquad \eta = i\frac{c^2}{a}$$

Solutions to this equation are the Parabolic Cylinder Functions (Whittaker and Watson) $D_\eta(z)$

two independent solutions are given by $D_{-\eta-1}(\pm iz)$

From the asymptotic behavior as $x \longrightarrow \infty$ one can extract the crossing probability

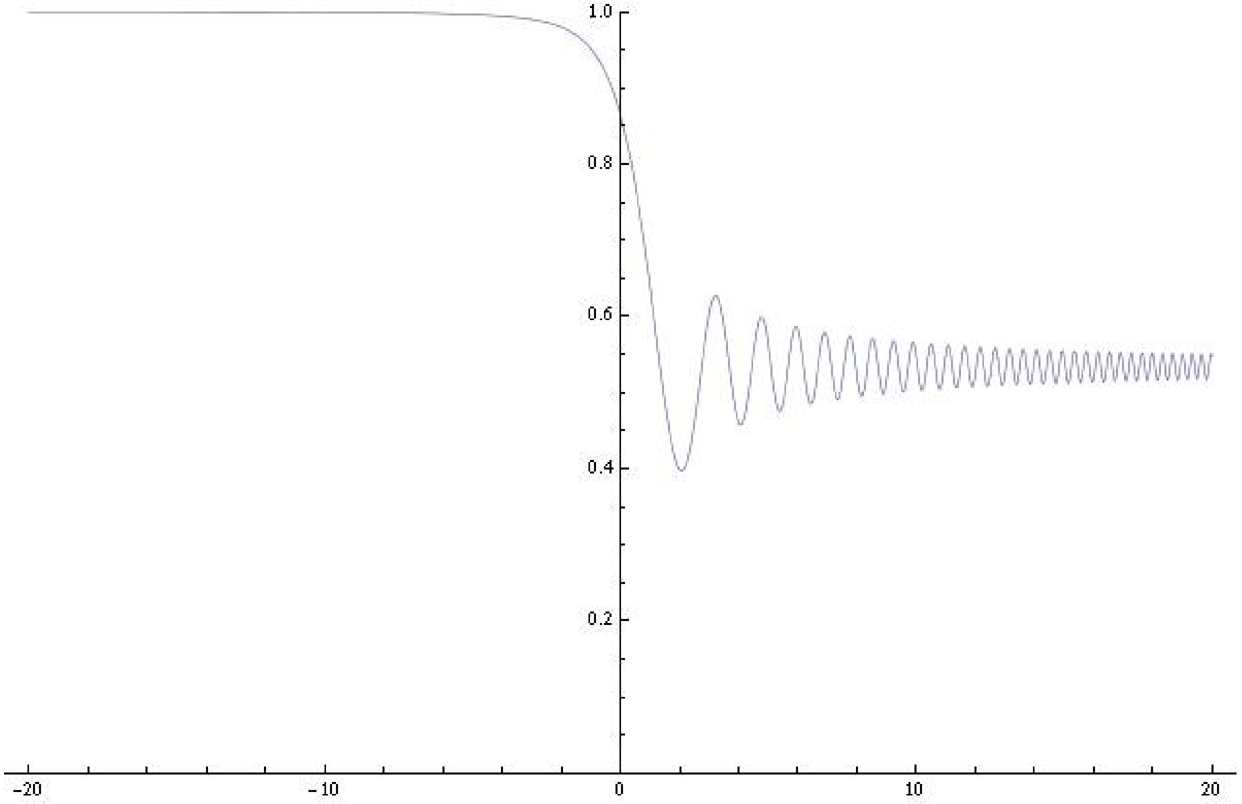
$$P_c = e^{-\frac{2\pi c^2}{a}}$$

In our case

$$P_c = \exp[-\pi \frac{\Delta m^2 \sin^2 2\theta}{4p\cos 2\theta|\frac{1}{n_e}\frac{dn_e}{dx}|_{res}}]$$

Example of non-adiabatic evolution

Probability vs. position



$$\eta = 0.1$$

$$\exp(-2\pi\eta) \simeq 0.53$$

Three generations

$$U = V_{23} V_{13} V_{12}$$

$$V_{12} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$c_{ij} = \cos\theta_{ij}$$

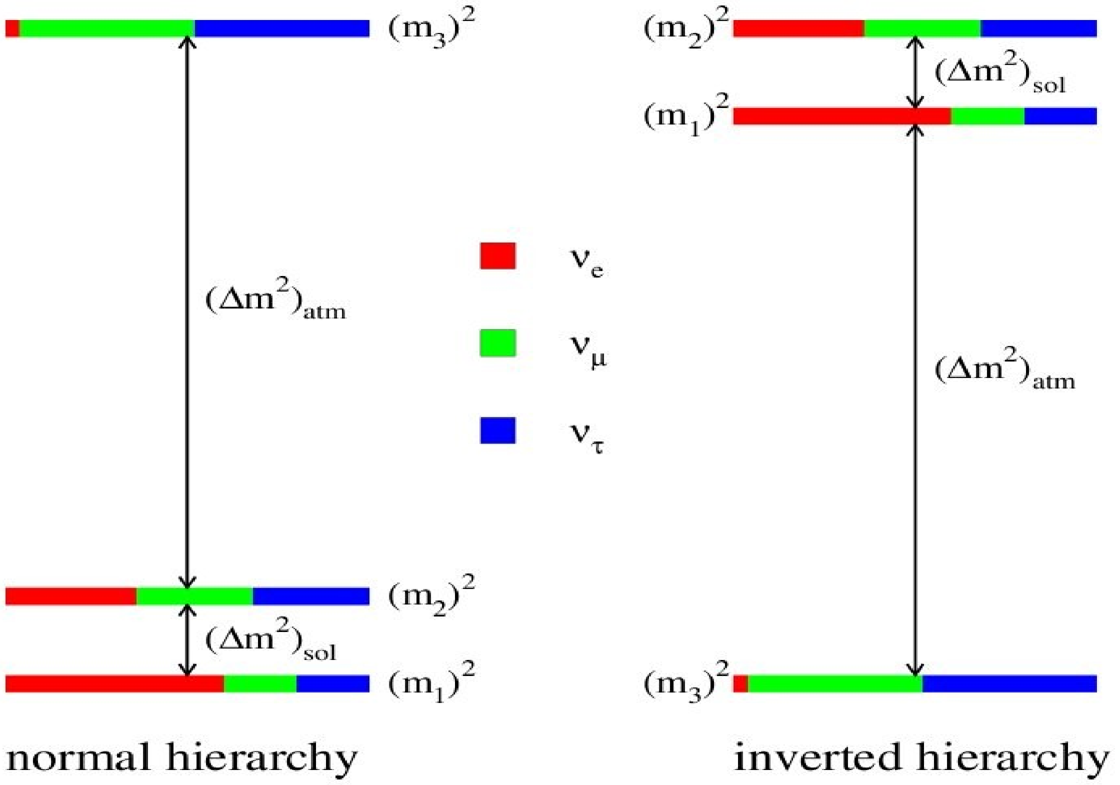
$$s_{ij} = \sin\theta_{ij}$$

$$V_{13} = \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix}$$

δ Is the CP violating phase

$$V_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}$$

Phenomenology



Evolution equation

$$i\frac{d}{dx}\begin{pmatrix} a_e \\ a_\mu \\ a_\tau \end{pmatrix} = H \begin{pmatrix} a_e \\ a_\mu \\ a_\tau \end{pmatrix}$$

$$H = UM^2U^\dagger + diag(V,0,0)$$

It can be reduced to an effective two-level system under the approximation

$$|\Delta m^2_{21}| \ll |\Delta m^2_{31}| \simeq |\Delta m^2_{32}|$$

Then $$M^2 \simeq diag(0, 0, \Delta m^2_{31}/2p)$$

By performing the transformation

$$\begin{pmatrix} a_e \\ a_\mu \\ a_\tau \end{pmatrix} = V_{23} \begin{pmatrix} a_a \\ a_b \\ a_c \end{pmatrix}$$

$|\nu_b >$ decouples, while the other two follow

$$i \frac{d}{dx} \begin{pmatrix} a_a \\ a_c \end{pmatrix} = \begin{pmatrix} \frac{\Delta m^2_{31}}{2p} s^2_{13} + V & \frac{\Delta m^2_{31}}{2p} s_{13} c_{13} e^{-i\delta} \\ \frac{\Delta m^2_{31}}{2p} s_{13} c_{13} e^{i\delta} & \frac{\Delta m^2_{31}}{2p} c^2_{13} \end{pmatrix} \begin{pmatrix} a_a \\ a_c \end{pmatrix}$$

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