Deep Inelastic Scattering and Neutrinos

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Literature

Elementary particle physics: Concepts and phenomena. By Otto Nachtmann. Springer, 1990. (In particular chapters 18 and 19.)

QCD and collider physics.
By R. K. Ellis , W. J. Stirling and B. R. Webber. Cambridge Univ. Pr., 1996. (In particular chapter 4.)

This lecture can be found at:
http://www.desy.de/~brandenb/dis-varenna.ps

I. Introduction and motivation

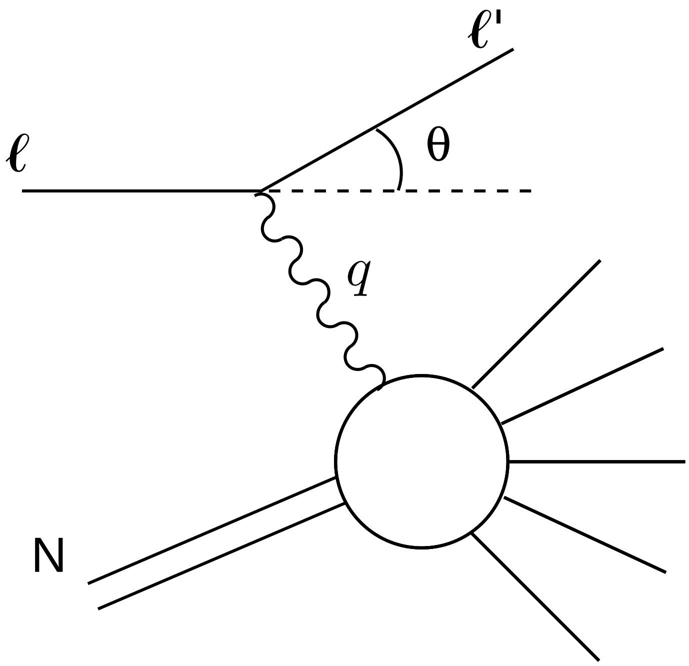
What is Deep Inelastic Scattering?

Generic reaction:

$\ell(k) + N(p) \to \ell'(k') + X(p')$,

$\ell, \ell' = e^{\pm}, \mu^{\pm}, \nu_\ell, \bar{\nu}_\ell,\; N = n, p, \ldots$

Kinematics:



$$Q^2 = -q^2 = -(k-k')^2 = 4EE' \sin^2 \theta/2$$

$$M^2 = p^2$$

$$\nu = p \cdot q = M(E - E')$$

$$x = Q^2/(2\nu)$$

$$y = (p \cdot q)/(p \cdot k) = 1 - E'/E \quad \text{('inelasticity')}$$

(θ, E, E' defined in the nucleon rest frame)

- Invariant mass of final state hadrons:
  $W^2 = p'^2 = (p+q)^2 = M^2 + 2\nu - Q^2 \geq M^2$
  (conservation of baryon number!)
- Deep $\leftrightarrow$ Q^2 large: $Q^2 \gg \Lambda^2_{\rm QCD} \approx (200\ \mathrm{MeV})^2$ to allow for QCD perturbation theory
  Spatial resolution $\sim 1/\sqrt{Q^2} = 2 \times 10^{-16}\mathrm{m}\ [\mathrm{GeV}]/\sqrt{Q^2}$
- Inelastic $\leftrightarrow$ $M^2 < W^2 \leftrightarrow x < 1$ (nucleon is destroyed)

I. Introduction and motivation

DIS and neutrinos

In general, we have for a neutrino detector

Counting rate = neutrino flux $\times$ cross section

- if one wants to know the neutrino flux, cross sections have to be computed theoretically
- if neutrino flux is known, counting rates can be used to test the theory used to compute cross sections

Where is DIS relevant? Some examples:

1. Solar neutrinos: Typical neutrino energies $\sim$ MeV $\Rightarrow$ DIS not applicable

2. Atmospheric neutrinos: Need cross section for

$$\nu_\ell(\bar{\nu}_\ell) + N \to \ell^\mp + X.$$

DIS cross section is used for $W > W_c = 1.4$ GeV.

P. Lipari, M. Lusignoli, F. Sartogo (1994)

This corresponds to constraint

$$2m_N E_\nu y(1-x) \geq W_c^2 - m_N^2,$$

with $y = 1 - E_\ell/E_\nu,\ x = Q^2/(2m_N E_\nu y)$.

Example: $E_\nu = 10$ GeV, $E_\mu = 5$ GeV, $m_N \approx 1$ GeV $\Rightarrow$ $y = 0.5,\ x \leq 0.9,\ Q^2 = 10\ \mathrm{GeV}^2 x$.

I. Introduction and motivation

Where is DIS relevant? Some examples:

3. Search for neutrino oscillations at NuTeV:

$$\nu_{e,\mu}(\bar{\nu}_{e,\mu}) + N \to \nu_{e,\mu}(\bar{\nu}_{e,\mu}) + X,$$
$$\nu_{e,\mu}(\bar{\nu}_{e,\mu}) + N \to e^{\mp}, \mu^{\mp} + X$$

Neutrinos have energies of up to 350 GeV (ν_μ) and 180 GeV (ν_e). Events are required to deposit at least 30 GeV in the calorimeter $\Rightarrow y \approx Q^2/(sx) > 0.09\ (0.16)$

4. Neutrino telescopes: (AMANDA, DUMAND II, BAIKAL, NESTOR, Pierre Auger Cosmic Ray Observatory, OWL, IceCube,. . .)

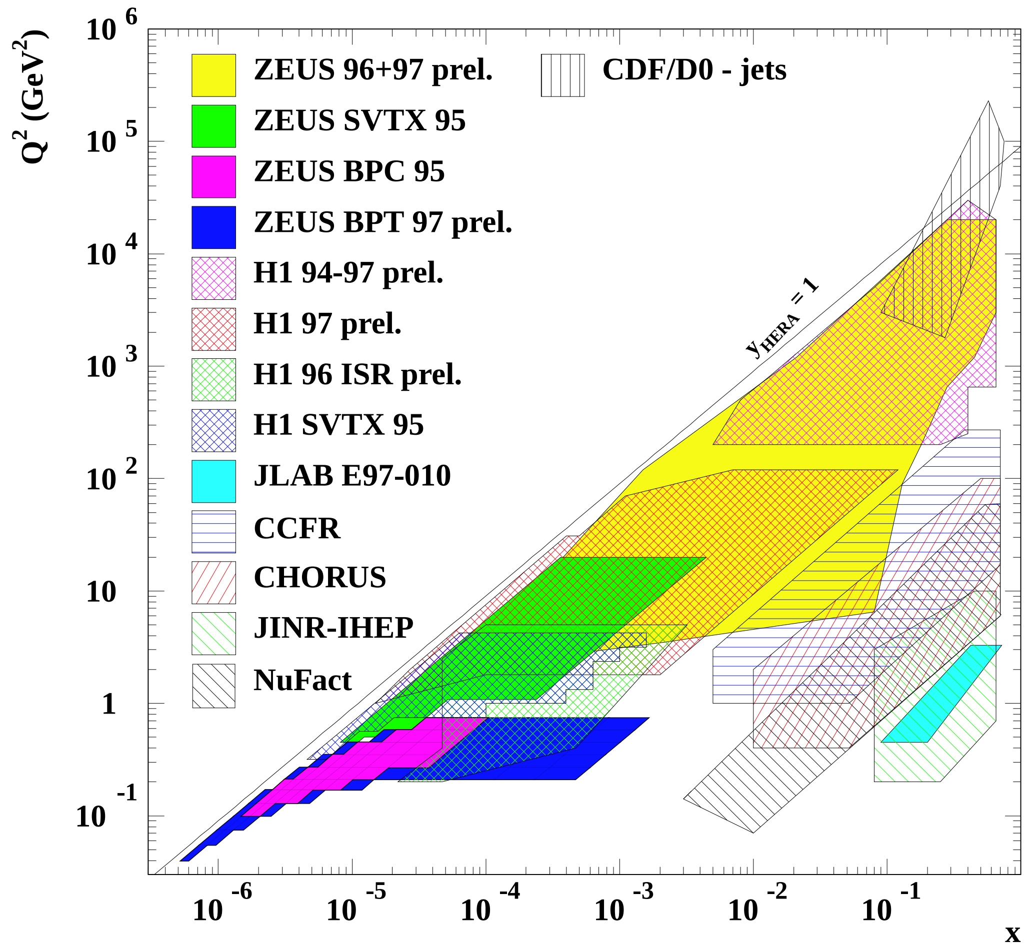
Ultra high energy neutrinos from ‘cosmic accelerators’: Energies up to 10^{21} eV. Detection of neutrino-induced muons:

$$\nu_\mu(\bar{\nu}_\mu) + N \to \mu^{\mp} + X$$

Example: $E_\nu = 10^{11}$ GeV, $E_\mu = 100$ TeV, $m_N \approx 1$ GeV $Q^2 = 200\ \mathrm{GeV}^2 \Rightarrow x = 10^{-9}$!

I. Introduction and motivation

Sensitivity of different experiments on regions in (x, Q^2) plane:



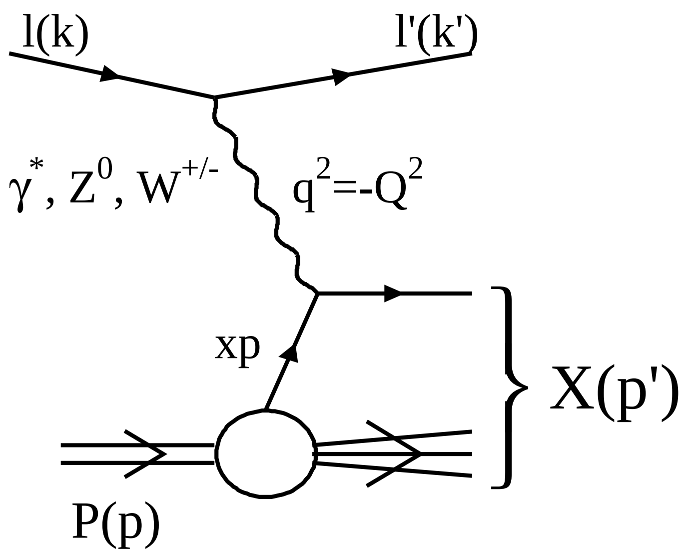
- HERA $(e + p \rightarrow e \text{ or } \nu + X)$ accesses kinematic regions of very small x and high $Q^2 > M_{W,Z}^2 \approx 10^4 \text{ GeV}^2 \Rightarrow$ DIS at electroweak energy scale, spatial resolution of up to 10^{-18} m $\sim \frac{r_{\text{proton}}}{1000}$

I. Introduction and motivation

Parton model:
DIS cross sections depend on quark and gluon content (Parton Distribution Functions) of the nucleon N.

PDFs are functions of x (interpreted as parton momentum fraction) and Q^2.

PDFs are assumed to be universal, i.e. can be extracted from some dedicated experiments and then used to compute cross sections for other processes.



Parton model was verified experimentally in 1969 by fixed target $eN \to eX$ deep inelastic scattering at SLAC (Nobel Prize 1990 for J.I. Friedman, H.W. Kendall, R.E. Taylor). Since then, numerous experiments.

CERN μN: EMC,BCDMS,NMC; νN: CDHSW, CHARM, CHORUS

FNAL μN: E-665; νN: CCFR, NuTeV (all fixed target)

DESY ep collider: H1, ZEUS

Further: polarized beams & targets (will not be discussed)

II. The parton model

Goal Extract Parton Distribution Functions from DIS experiments

Consider first the reaction

$$e^{\pm} + p \to e^{\pm} + X$$

for the case that the exchanged boson is a photon. At high energies, also the exchange of a Z boson becomes relevant. Further, if one of the leptons is a neutrino, a W boson can be exchanged, see below.

Differential cross section (for $E \gg M$):

$$\frac{d^2\sigma^{\text{e.m.}}}{dx dQ^2} = \frac{4\pi\alpha^2_{\text{e.m.}}}{Q^4} \left\{ \left[1 + (1-y)^2\right] F_1^{\text{e.m.}} + \frac{1-y}{x} \left(F_2^{\text{e.m.}} - 2x F_1^{\text{e.m.}}\right) \right\} \qquad (*)$$

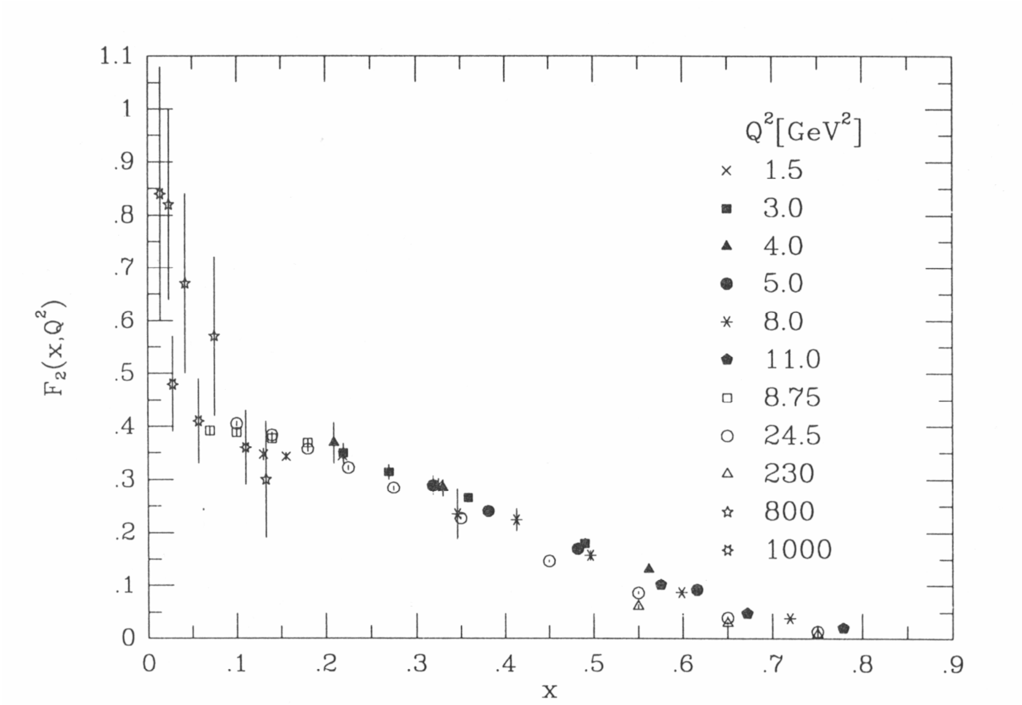
- $F_{1,2}(x, Q^2)$ are called structure functions of the proton.
- The Bjorken limit is defined as follows:

$$\boxed{Q^2, \nu \to \infty, \quad x \text{ fixed}}$$

II. The parton model

In the Bjorken limit the structure functions obey to good approximation a scaling law, i.e. they only depend on the dimensionless variable x:

F_2 as a function of x for different values of Q^2



Bjorken scale-invariance implies that the virtual photon is scattered off pointlike constituents, since otherwise the structure functions would depend also on Q/Q_0, where Q_0 characterizes the size of the constituents.

II. The parton model

Derivation of the cross section:

$$\sigma^{ep\to eX} = \frac{1}{4ME}\sum_X \int d\Phi (2\pi)^4 \delta(q+p-p_X) \times \frac{1}{4}\sum_{\text{spins}} |T_{fi}|^2$$

Amplitude:

$$iT_{fi} = -ie\bar{u}(k')\gamma^\mu u(k)\frac{1}{q^2}\langle X(p_X)|j_\mu(0)|P(p)\rangle,$$

ej_μ: hadronic part of the e.m. current. Divide phase space and $|T_{fi}|^2$ in leptonic and hadronic part:

$$d\Phi = \frac{d^3k'}{(2\pi)^3 2E'} d\Phi_X = \frac{1}{16\pi^2}\frac{y}{x} dx dQ^2 d\Phi_X$$

$$\frac{1}{4}\sum_{\text{spins}} |T_{fi}|^2 \equiv \frac{e^4}{Q^4} L^{\mu\nu} h_{\mu\nu}^{\text{e.m.}}$$

Lepton tensor:

$$L^{\mu\nu} = \frac{1}{4}\sum_{\text{spins}} [\bar{u}(k')\gamma^\mu u(k)] [\bar{u}(k)\gamma^\nu u(k')] = \frac{1}{4}\text{Tr}\,[\not{k}'\gamma^\mu \not{k}\gamma^\nu] = k^\mu k'^\nu + k^\nu k'^\mu - g^{\mu\nu} kk'$$

Hadron tensor:

$$H_{\mu\nu}^{\text{e.m.}} \equiv \sum_X \int d\Phi_X (2\pi)^4 \delta(q + p - p_X) h_{\mu\nu}^{\text{e.m.}}$$

$$= \sum_X \int d\Phi_X (2\pi)^4 \delta(q + p - p_X) \langle P(p)|j_\mu(0)|X(p_X)\rangle\langle X(p_X)|j_\nu(0)|P(p)\rangle$$

The differential cross section now reads:

$$\frac{d^2\sigma}{dx dQ^2} = \frac{1}{4ME} \frac{\alpha_{\text{e.m.}}^2}{Q^4} \frac{y}{x} L^{\mu\nu} H_{\mu\nu}^{\text{e.m.}} \qquad (\dagger)$$

The (e.m.) hadron tensor can be parametrized by two structure functions due to Lorentz covariance, parity invariance and e.m. current conservation ($q^\mu H_{\mu\nu} = 0$):

$$H_{\mu\nu}^{\text{e.m.}} = 8\pi \left[\left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) F_1^{\text{e.m.}}(x, Q^2) + \frac{2x}{Q^2} \left(p_\mu + \frac{q_\mu}{2x} \right) \left(p_\nu + \frac{q_\nu}{2x} \right) F_2^{\text{e.m.}}(x, Q^2) \right]$$

Inserting this into (†) gives the result (*) (use $ME = Q^2/(2xy)$).

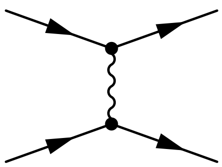
II. The parton model

The naive parton model

- The parton model can be formulated most easily in the infinite momentum frame (proton moves very fast),

$$\text{proton}: \qquad P^\mu = (P, 0, 0, P), \qquad P \gg M.$$

- First consider the scattering process $e^- q \to e^- q$.



$$\frac{d\hat{\sigma}}{dQ^2} = \frac{2\pi\alpha_{\text{e.m.}}^2 Q_q^2}{Q^4}\left[1 + (1-y)^2\right]$$

- Define ξ to be momentum fraction carried by the quark:

$$\text{quark}: \qquad p_q^\mu = \xi(P, 0, 0, P), \qquad P \gg M.$$

- The on-shell condition for the scattered quark reads:

$$(p_q')^2 = (p_q + q)^2 = q^2 + 2p_q q = -2Pq(x - \xi)$$

- By inserting $1 = \int_0^1 dx \delta(x - \xi)$ we get for the double differential cross section for electron-quark scattering:

$$\frac{d^2\hat{\sigma}}{dx dQ^2} = \frac{4\pi\alpha_{\text{e.m.}}^2}{Q^4}\left[1 + (1-y)^2\right]\frac{1}{2}Q_q^2\delta(x - \xi)$$

Therefore, the structure functions for a quark with momentum fraction ξ are

$$\hat{F}_2^\xi = 2x\hat{F}_1^\xi = xQ_q^2\delta(x-\xi)$$

The structure function of the proton obviously is no δ-function, i.e. quarks can carry different momentum fractions ξ. We can formulate the parton model as follows:

- $q(\xi)d\xi$ is the probability that a quark carries a momentum fraction of the proton between ξ and $\xi + d\xi$, $0 \leq \xi \leq 1$.
- The virtual photon scatters incoherently off the quark constituents.

The proton structure function is obtained from the quark structure function by weighting the latter with the quark probability distribution $q(\xi)$:

$$\begin{aligned} F_2(x) = 2xF_1(x) &= \sum_{q,\bar{q}} \int_0^1 d\xi q(\xi) x Q_q^2 \delta(x-\xi) \\ &= \sum_{q,\bar{q}} Q_q^2 x q(x) \end{aligned}$$

II. The parton model

- The previous result can be slightly rewritten in terms of a folding between the quark distribution q and the quark structure function at $\xi = 1$:

$$F_2(x) = 2xF_1(x) = \sum_{q,\bar{q}} \int_x^1 \frac{dy}{y} q(y) \hat{F}_2(x/y),$$

$$\hat{F}_2(x) \equiv Q_q^2 x \delta(1-x)$$

- $F_2(x, Q^2) = F_2(x)$: parton model gives Bjorken scaling
- The result $F_2(x) = 2xF_1(x)$ is known as the Callan-Gross relation. It follows from the spin-1/2 nature of the quarks. One calls

$$F_L(x, Q^2) = F_2 - 2xF_1$$

the longitudinal structure function, $F_L = O(\alpha_s)$.
(*) can also be written as follows:

$$\frac{d^2\sigma^{\rm e.m.}}{dx dQ^2} = \frac{2\pi\alpha_{\rm e.m.}^2}{xQ^4} \left\{ \left[1 + (1-y)^2\right] F_2^{\rm e.m.} - y^2 F_L^{\rm e.m.} \right\}$$

II. The parton model

Explicitly, we have (at tree level and neglecting b quarks) for a proton target:

$$F_2^{\mathrm{e.m.}} = x\left[\frac{4}{9}(u+\bar{u}+c+\bar{c})+\frac{1}{9}(d+\bar{d}+s+\bar{s})\right]$$

How can the different quark distributions be disentangled? Analyse DIS reactions for different initial and final states!

1. Neutrino-nucleon scattering

$$\nu_\ell, \bar{\nu}_\ell + N \to \ell^\mp + X$$

A new structure function F_3 appears due to the parity violating charged current interaction. (Both in $L_{\mu\nu}$ and $H_{\mu\nu}$ a Levi-Cevita tensor is present.)

$$\frac{d^2\sigma^{\nu,\bar{\nu}N}}{dxdQ^2} = \frac{G_F^2}{2\pi x}\left(\frac{m_W^2}{Q^2+m_W^2}\right)^2 \phi^{\nu,\bar{\nu}N}$$

$$\phi^{\nu,\bar{\nu}N} = \frac{1}{2}\left(Y_+F_2^{\nu,\bar{\nu}N} - y^2F_L^{\nu,\bar{\nu}N} \pm xY_-F_3^{\nu,\bar{\nu}N}\right),$$

$$Y_\pm = 1 \pm (1-y)^2$$

II. The parton model

Parton model results: $F_L^{\nu,\bar{\nu}N} = O(\alpha_s)$,

$$F_2^{\nu p} = 2x\Big[d + s + \bar{u} + \bar{c}\Big]; \quad F_2^{\bar{\nu}p} = 2x\Big[u + c + \bar{d} + \bar{s}\Big]$$

$$xF_3^{\nu p} = 2x\Big[d + s - \bar{u} - \bar{c}\Big]; \quad xF_3^{\bar{\nu}p} = 2x\Big[u + c - \bar{d} - \bar{s}\Big]$$

$u, d, \ldots$ are the parton distributions in the proton. Isospin symmetry yields

$$N_d{}^n = N_u{}^p \equiv u, \;\; N_u{}^n = N_d{}^p \equiv d, \;\; \text{etc}$$

In particular one gets for an isoscalar target N, neglecting antiquarks and 'sea' quarks:

$$F_2^{\nu N} = \frac{1}{2}(F_2^{\nu p} + F_2^{\nu n}) \approx x(u + d),$$

For $eN \to \gamma^* \to eX$ DIS:

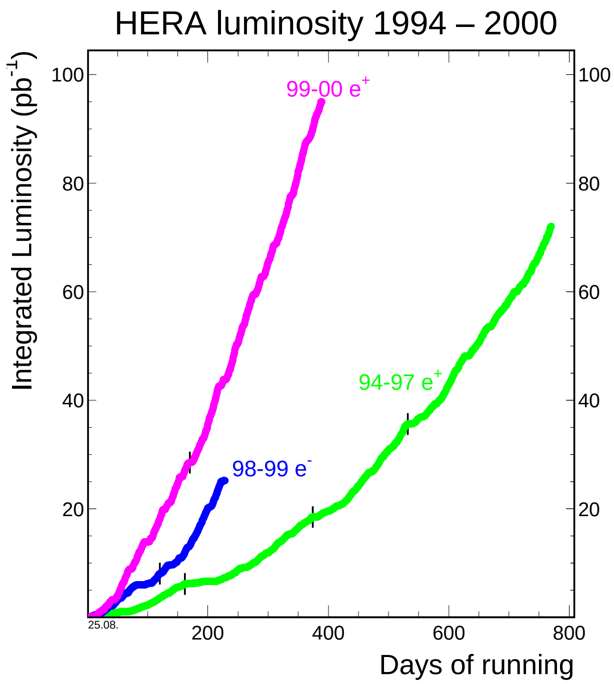
$$F_2^{eN} = \frac{1}{2}(F_2^{ep} + F_2^{en}) \approx \frac{5}{18}x(u + d)$$

$\Rightarrow$

$$\frac{F_2^{eN}}{F_2^{\nu N}} \approx \frac{5}{18}$$

in good agreement with experiment.

HERA results



HERA: $e^{\pm}p$ collisions

at $\sqrt{s} = 300/318$ GeV

$e^{\pm}$ energy: 27.5 GeV

p energy: 820/920 GeV

(before/after 1997)

2. Charged current interaction at HERA

$$e^{\pm} + p \to \bar{\nu}_e, \nu_e + X$$

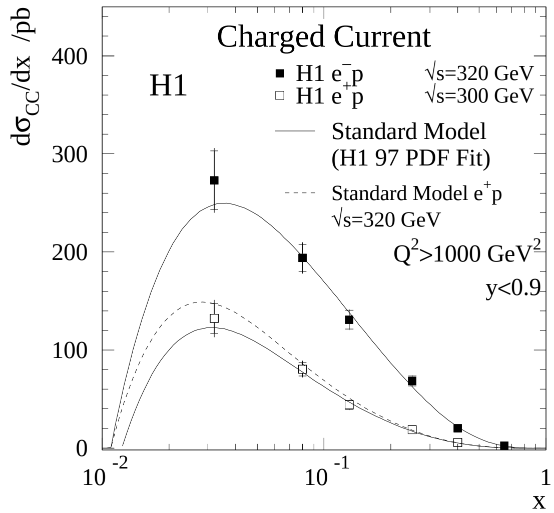
$$\frac{d^2\sigma^{\pm}_{CC}}{dx dQ^2} = \frac{G_F^2}{2\pi x}\left(\frac{m_W^2}{Q^2 + m_W^2}\right)^2 \phi^{\pm}_{CC}$$

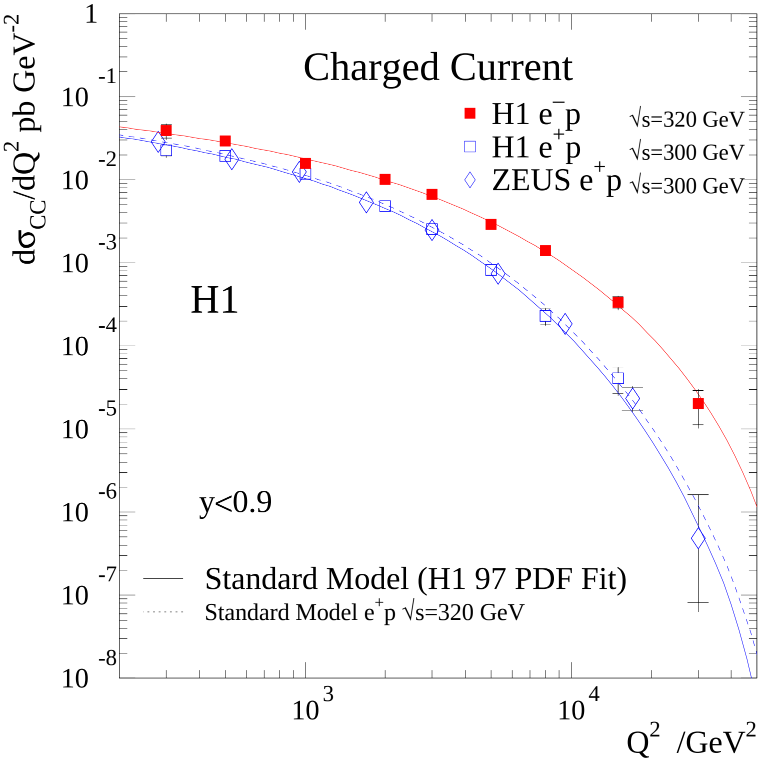
Parton model results (neglecting b and t and quark mixing) at tree level for *unpolarized* $e^{\pm}$:

$$\phi^{+}_{CC} = x\left[(d+s)(1-y)^2 + (\bar{u}+\bar{c})\right]$$

$$\phi^{-}_{CC} = x\left[(u+c) + (\bar{d}+\bar{s})(1-y)^2\right]$$

HERA results on $d\sigma_{CC}/dx,\ d\sigma_{CC}/dQ^2$





Q^2 dependence:

PDF values in integration region $x \in [Q^2/s, 1]$

PDFs are weighted with $y = Q^2/(xs)$-dependent factors

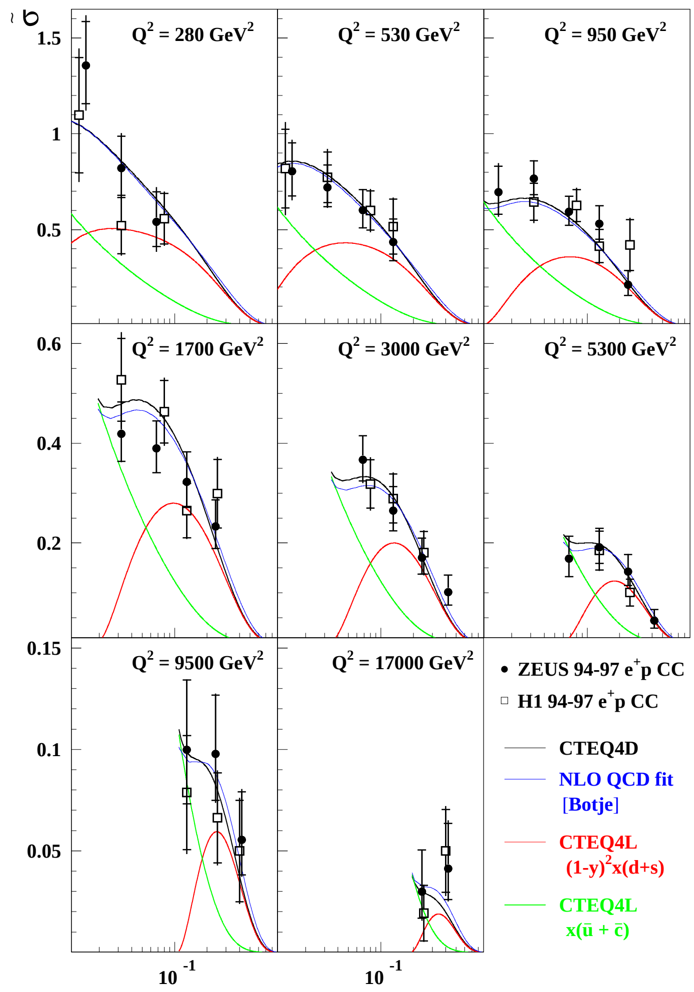
W propagator $\sim \left[m_W^2/(Q^2 + m_W^2)\right]^2$

HERA results on $d\sigma^{\pm}_{CC}/dx/dQ^2$

Define 'reduced CC cross section'

$$\tilde{\sigma}^{\pm}_{CC} \equiv \frac{2\pi x}{G_F^2}\left(\frac{m_W^2+Q^2}{m_W^2}\right)^2 \frac{d^2\sigma^{\pm}_{CC}}{dx\,dQ^2} = \phi^{\pm}_{CC}$$

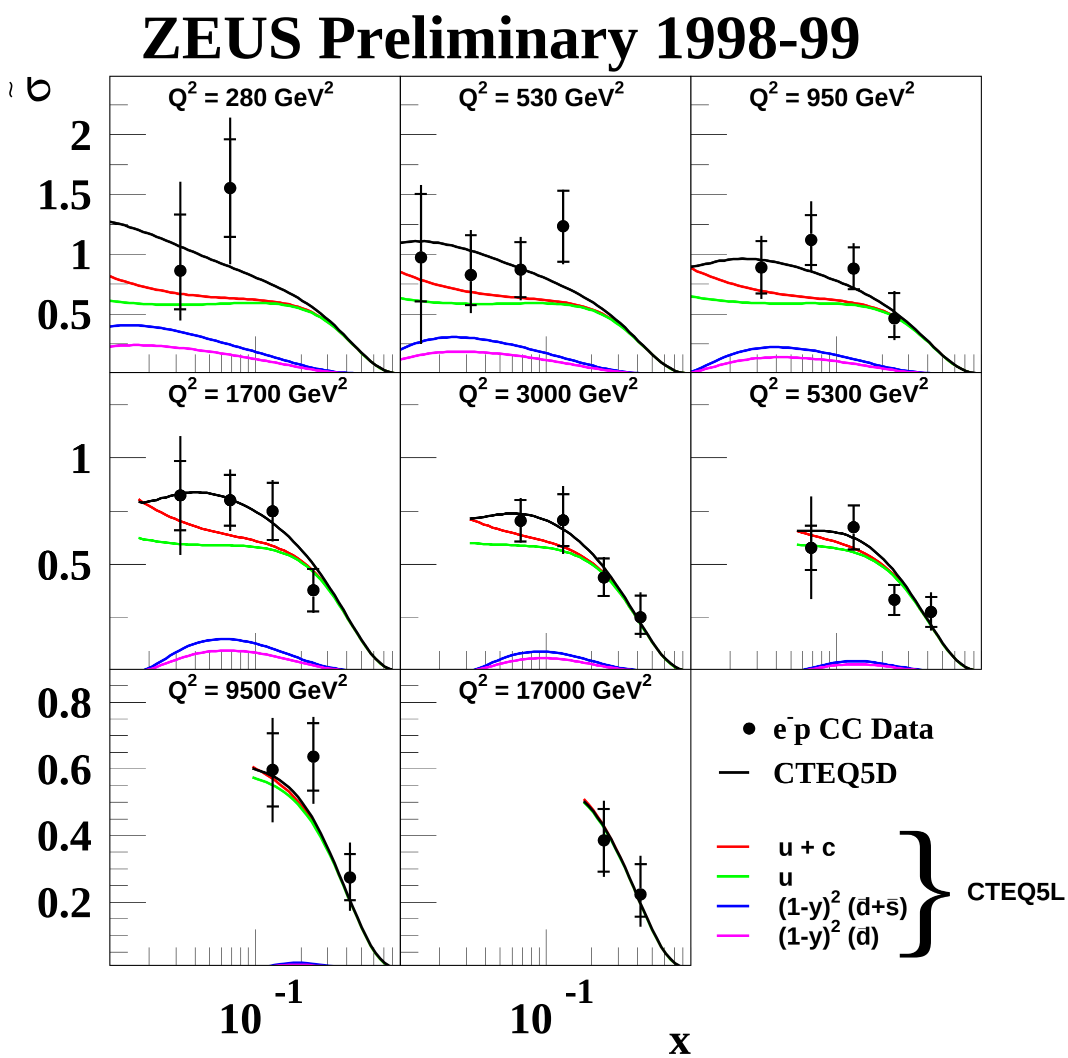
$$\phi^{+}_{CC} = x(d+s)(1-y)^2 + x(\bar{u}+\bar{c})$$



- $x > 0.2$: $(1-y)^2(d+s)$ dominates

HERA results on $d\sigma^{\pm}_{CC}/dx/dQ^2$

$$\tilde{\sigma}^{-}_{CC} \equiv \phi^{-}_{CC} = x(u+c) + x(\bar{d}+\bar{s})(1-y)^2$$



- $x > 0.2$: u dominates (no $(1-y)^2$ supression)
- $x < 0.1$: significant contribution from c and $\bar{s}$

2. Neutral current interaction at HERA

$$e^{\pm} + p \to e^{\pm} + X$$

$$\frac{d^2\sigma^{\pm}_{NC}}{dx dQ^2} = \frac{2\pi\alpha^2}{xQ^4}\left[Y_+\tilde{F}_2 - y^2\tilde{F}_L \mp xY_-\tilde{F}_3\right]$$

$$\tilde{F}_{2,L} \equiv F^{\mathrm{e.m.}}_{2,L} - v_e\chi_Z F^{\gamma Z}_{2,L} + (v_e^2 + a_e^2)\chi_Z^2 F^{Z}_{2,L},$$

$$x\tilde{F}_3 \equiv -a_e\chi_Z xF_3^{\gamma Z} + 2v_e a_e \chi_Z^2 xF_3^Z,$$

where v_e (a_e) is the vector (axial) coupling of the electron to the Z and

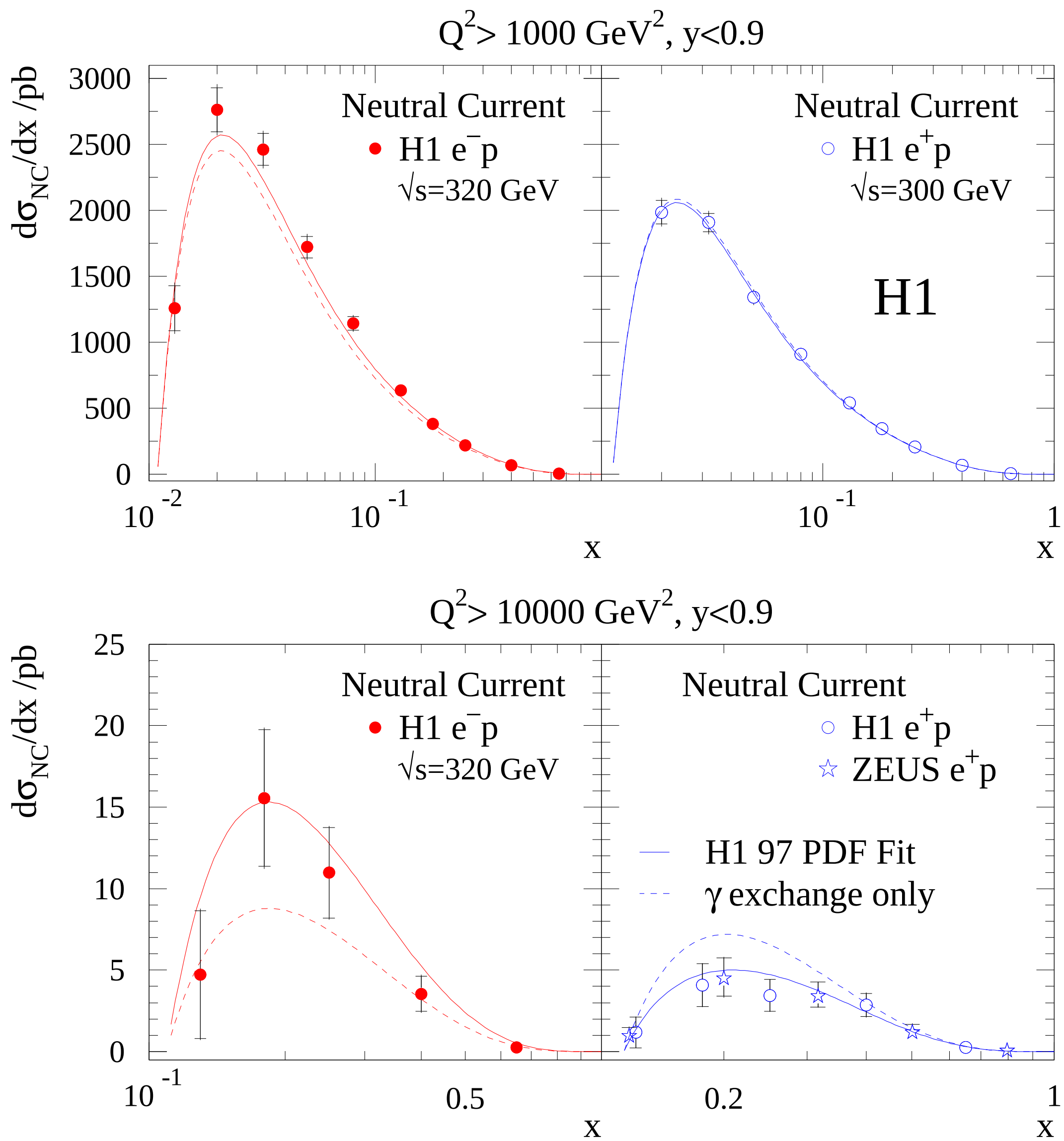
$$\chi_Z = \frac{\sqrt{2}G_F Q^2 m_Z^2}{4\pi\alpha(Q^2 + m_Z^2)}.$$

Parton model result at order α_s^0: $\tilde{F}_L = 0$,

$$\left[F_2^{\mathrm{e.m.}}, F_2^{\gamma Z}, F_2^Z\right] = x\sum_q \left[e_q^2, 2e_q v_q, v_q^2 + a_q^2\right]\{q + \bar{q}\},$$

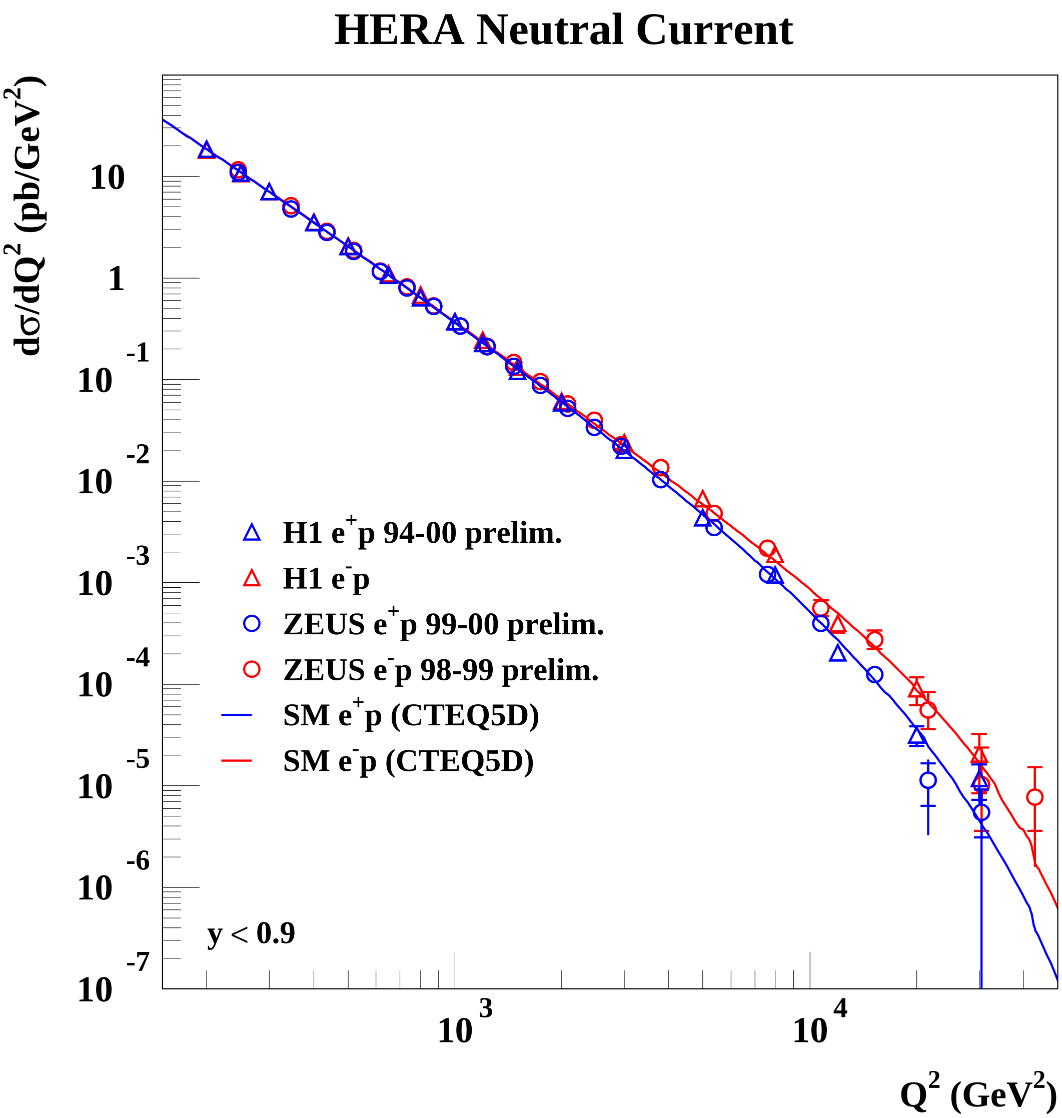
$$\left[xF_3^{\gamma Z}, xF_3^Z\right] = x\sum_q \left[2e_q a_q, 2v_q a_q\right]\{q - \bar{q}\}$$

HERA results on $d\sigma^{\pm}_{NC}/dx$



- dashed lines: contribution from γ exchange only ($m_Z \rightarrow \infty$) ($\sigma^{\gamma}(e^{-}p) > \sigma^{\gamma}(e^{+}p)$ due to different s)
  $\Rightarrow$ data is sensitive to Z exchange

HERA results on $d\sigma^{\pm}_{NC}/dQ^2$



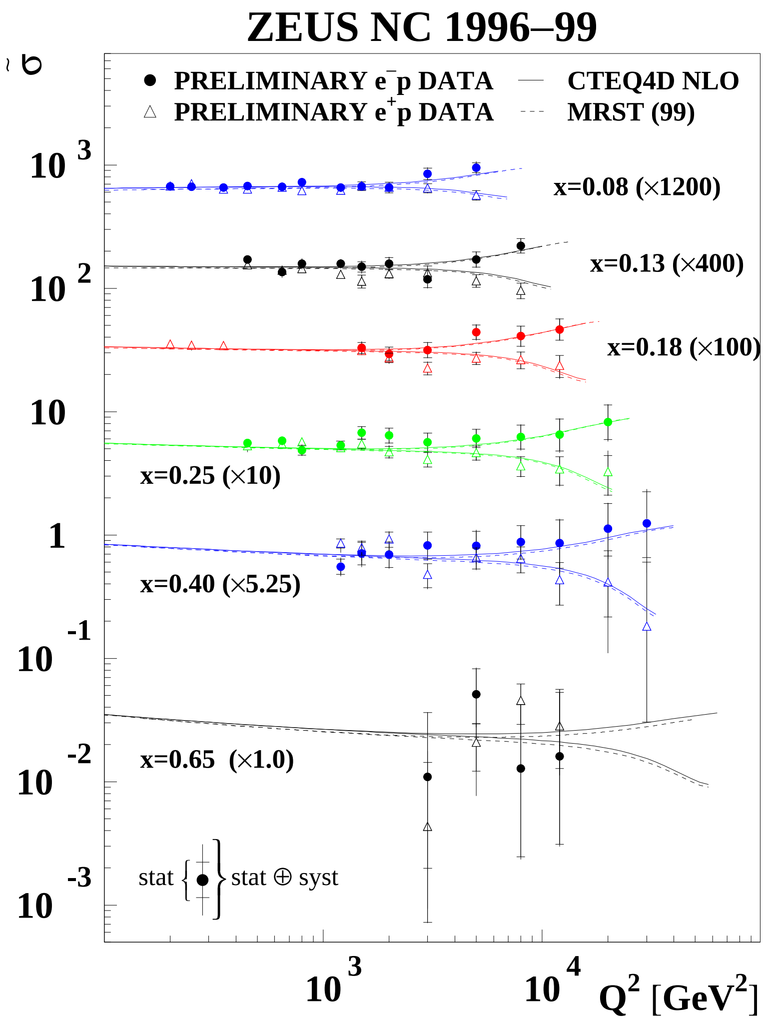
- difference $\frac{d\sigma}{dQ^2}(e^-p) - \frac{d\sigma}{dQ^2}(e^+p)$ caused by $x\tilde{F}_3$ term (Z exchange)

  e^+p data are corrected to $\sqrt{s} = 318$ GeV

HERA results on $d\sigma^{\pm}_{NC}/dx/dQ^2$

Reduced NC cross section

$$\tilde{\sigma}^{\pm}_{NC} \equiv \frac{xQ^4}{2\pi\alpha^2\left[1+(1-y)^2\right]}\frac{d^2\sigma^{\pm}_{NC}}{dxdQ^2}$$



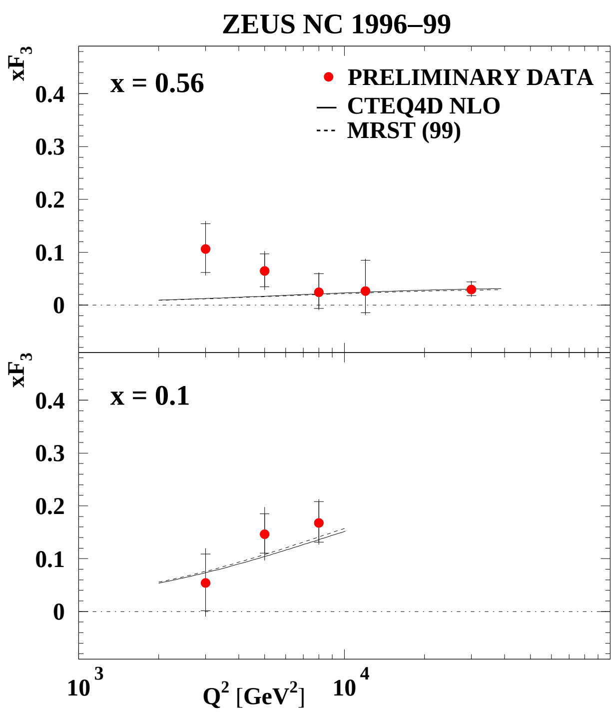
- for $Q^2 \ll m_Z^2$: $\tilde{\sigma}^{+}_{NC} \approx \tilde{\sigma}^{-}_{NC}$
- for $Q^2 \sim m_Z^2$: significant $x\tilde{F}_3$ terms

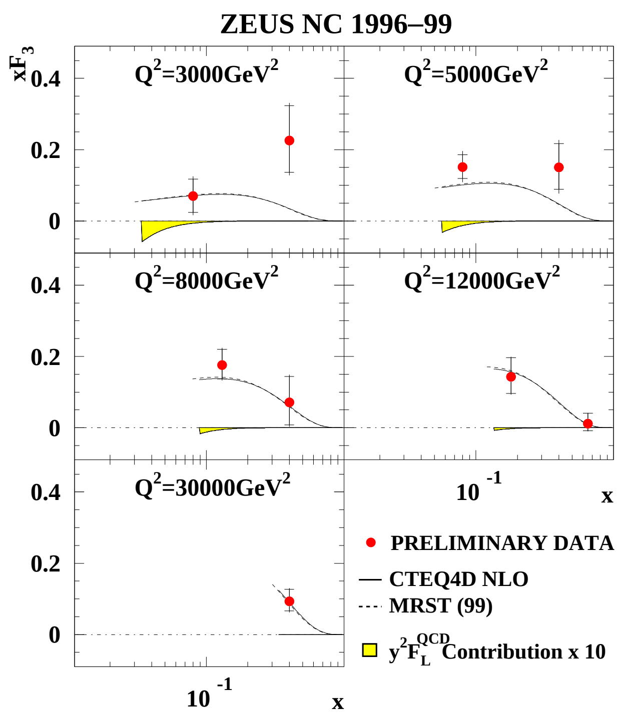
HERA results on xF_3

$x\tilde{F}_3$ is computed from

$$\tilde{\sigma}^-_{NC} - \tilde{\sigma}^+_{NC} = x\tilde{F}_3 \left[\frac{Y_{-\ 318}}{Y_{+\ 318}} + \frac{Y_{-\ 300}}{Y_{+\ 300}} \right] - \tilde{F}_L \left[\frac{y^2_{318}}{Y_{+\ 318}} - \frac{y^2_{300}}{Y_{+\ 300}} \right]$$

where $Y_{\pm\ \sqrt{s}} = 1 \pm (1-y)^2$ with $y = Q^2/(xs)$.



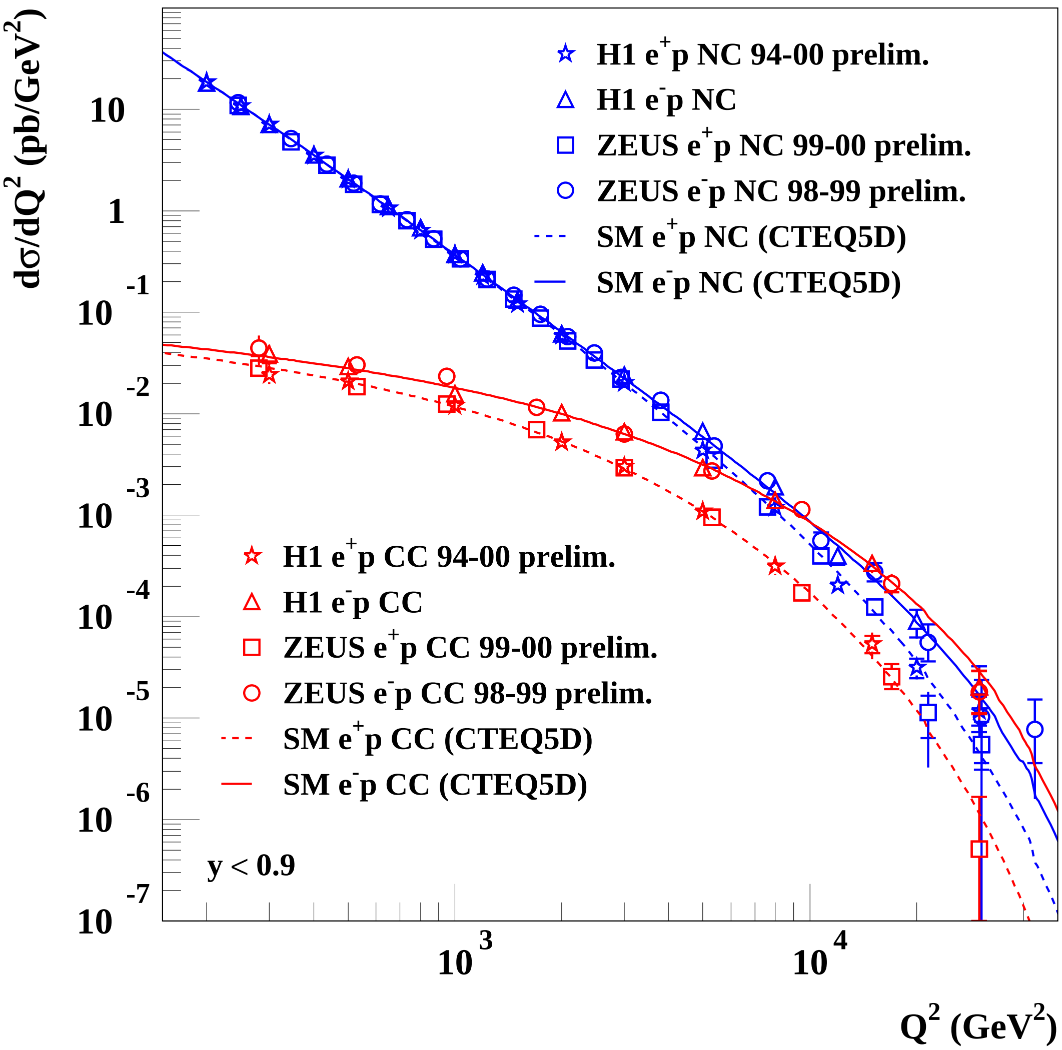


- $x\tilde{F}_3 \sim q - \bar{q}$, i.e. sensitive to u, d *valence* quark distributions

$(q = q_{\text{valence}} + q_{\text{sea}},\ q_{\text{sea}} = \bar{q}_{\text{sea}},\ c_{\text{valence}} = s_{\text{valence}} = 0)$

- Q^2 dependence is dominated by Z propagator term

NC versus CC



$\frac{d\sigma^{CC}}{dQ^2} \approx \frac{d\sigma^{NC}}{dQ^2}$ at very large Q^2: 'electroweak unification'

II. The parton model

Sum rules

Integrals over certain combinations of structure functions have particular values in the parton model. Such integrals are called sum rules and serve as a test of the parton model and constrain parametrizations for the PDFs. Examples:

- Momentum sum rule

$$\int_0^1 dx \left(9F_2^{eN} - \frac{3}{2}F_2^{\nu N}\right) = \sum_q \int_0^1 dx\ x\,(q+\bar{q}) \approx 0.5$$

$\Rightarrow$ only half of the proton's momentum is carried by the quarks! The rest is carried by gluons with momentum distribution $g(x)$. (The gluon distribution is not easily accesible in ep scattering, since the gluon does not couple directly to the photon.)

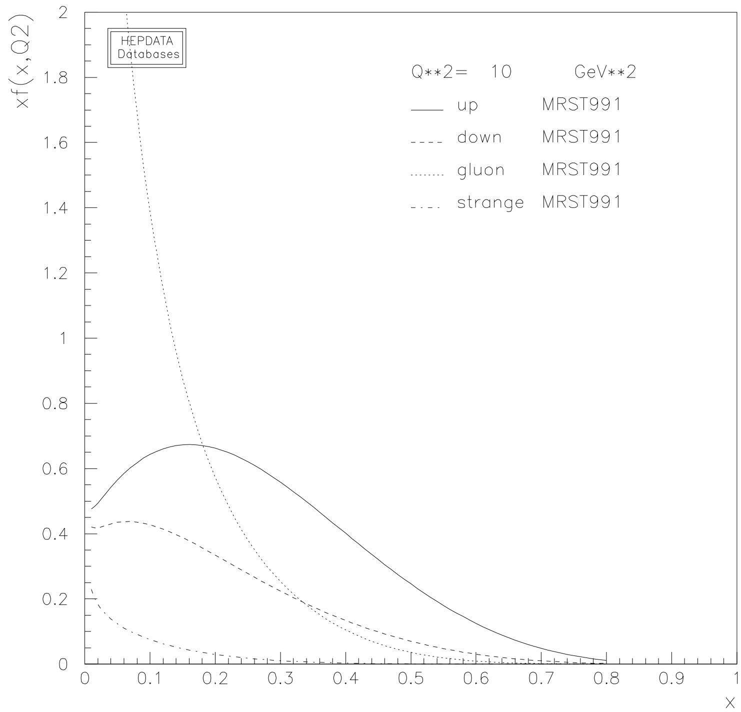
- Gross-Llewellyn Smith sum rule

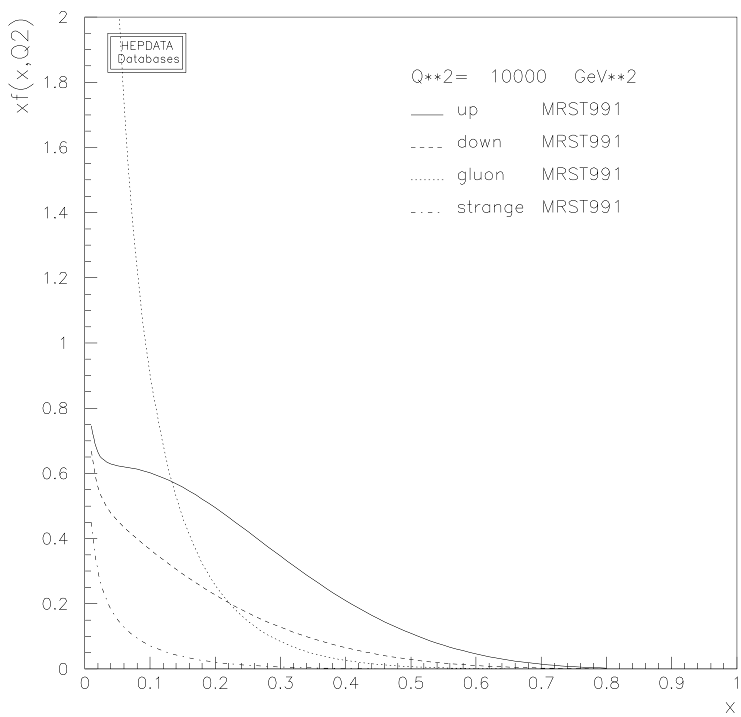
$$\int_0^1 dx F_3^{\nu N} = \frac{1}{2}\int_0^1 dx\left(F_3^{\nu p} + F_3^{\bar{\nu}p}\right)$$

$$= \int_0^1 dx(d - \bar{u} + u - \bar{d}) = \int_0^1 dx(u_{\rm v} + d_{\rm v}) = 3 + O(\alpha_s)$$

II. The parton model

Parton distribution functions are obtained by fitting an *ansatz* to all available data (DIS, Drell-Yan,. . .). (For details, see below.)





III. Scaling violation and DGLAP equations

Violation of Bjorken scaling and QCD

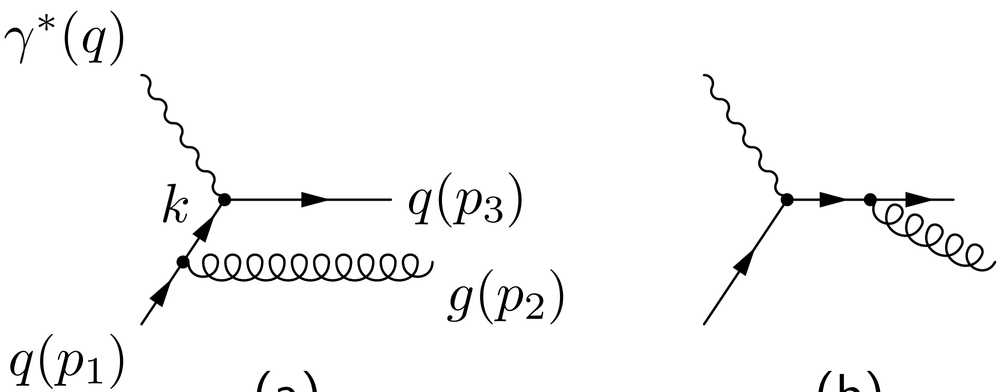
Closer inspection of the data shows that Bjorken scaling is not exact. QCD effects break scale invariance by inducing a dependence of the structure functions $\sim \ln(Q^2)$.

Qualitative explanation:

- In our derivation of the parton model result for F_i we assumed that the momentum of the quark that is struck by the photon is parallel to the proton momentum. This assumption is invalid if the quark emits a hard gluon!
- The quark can thus aquire a large transverse momentum k_T. The probability for this is $\sim \alpha_s dk_T^2/k_T^2$, and integration up to the kinematic limit $k_T^2 = Q^2$ gives terms $\sim \alpha_s \ln(Q^2)$.

Quantitatve description of scaling violation

We consider now the first order QCD corrections to the process $q\gamma^* \to q$ (i.e. the trivial leptonic part is separated off). There are two diagrams involving real gluon emission:



(a) (b)

Consider diagram (a):

- Lorentz invariant phase space:

$$\int \frac{d^4p_2}{(2\pi)^3}\frac{d^4p_3}{(2\pi)^3}\delta^+(p_2^2)\delta^+(p_3^2)(2\pi)^4\delta^4(p_1+q-p_2-p_3)$$

$$= \int \frac{d^4k}{4\pi^2}\delta^+((k+q)^2)\delta^+((p_1-k)^2)$$

III. Scaling violation and DGLAP equations

- Squared matrix element:

$$\frac{1}{2}\frac{1}{N_C}\sum_{\text{spins,colour}} |M|^2_{\mu\nu}$$

$$= \frac{1}{2}Q_q^2 g^2 C_F \frac{1}{k^4} \sum_{\text{gluon pol.}} \text{Tr}\left[(\not k + \not q)\gamma_\mu \not k \not\epsilon \not p_1 \not\epsilon^* \not k \gamma_\nu\right]$$

- The contribution to the quark structure function $\hat{F}_2$ contains an divergent term $\sim \int_0^{Q^2/x} d|k^2|/|k^2|$ corresponding to the case when the quark propagator goes on-shell.
- We regulate this by requiring $|k^2| > \kappa^2$. Inclusion of the virtual corrections (and diagram (b)) leads to the result:

$$\hat{F}_2(x, Q^2) = Q_q^2 x \left[\delta(1-x) + \frac{\alpha_s}{2\pi}\left(P^{qq}(x)\ln\frac{Q^2}{\kappa^2} + C(x)\right)\right]$$

with the quark splitting function

$$P^{qq}(x) = C_F\left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x)\right]$$

and $\int_0^1 dx \frac{f(x)}{(1-x)_+} = \int_0^1 dx \frac{f(x)-f(1)}{1-x}$. $C(x)$ is a calculable, finite function independent of Q^2.

Remarks

- In the quark structure function, the divergencies from real and virtual gluons do not cancel!
- Our method to regulate the divergence was completely adhoc. If we had used dimensional regularization ($d = 4 + 2\epsilon$), the divergent piece would be the same with $\ln(Q^2/\kappa^2) \to 1/\epsilon$.

  What does this divergence mean?

- It is a collinear divergence (q is parallel to g).
- It is not cancelled by the virtual corrections, since the photon can distinguish between a quark and a quark-gluon pair with the same overall momentum and the Kinoshita-Lee-Nauenberg theorem does not apply
- However, the quark structure function is no observable quantity. The proton structure function is obtained by folding $\hat{F}_2$ with the quark distribution:

$$F_2(x, Q^2) = x \sum_{q,\bar{q}} Q_q^2 \Bigg[q_0(x) + \frac{\alpha_s}{2\pi} \int_x^1 \frac{d\xi}{\xi} q_0(\xi) \left\{ P^{qq}\left(\frac{x}{\xi}\right) \ln \frac{Q^2}{\kappa^2} + C\left(\frac{x}{\xi}\right) \right\} + \ldots \Bigg]$$

III. Scaling violation and DGLAP equations

- q_0 can be interpreted as a bare distribution (analogy to 'bare' coupling and masses) and divergent parts of $\hat{F}_2$ can be absorbed into the quark distribution by defining a renormalized quark distribution:

$$q(x,\mu_F^2) = q_0(x) + \frac{\alpha_s}{2\pi}\int_x^1 \frac{d\xi}{\xi}$$

$$\times\, q_0(\xi)\left\{P^{qq}\left(\frac{x}{\xi}\right)\ln\frac{\mu_F^2}{\kappa^2} + (C - C_q)\left(\frac{x}{\xi}\right)\right\}$$

- μ_F is the factorization scale and C_q is an arbitrary finite term that reflects the scheme dependence of the definition of $q(x,\mu_F^2)$.
- The proton structure function to order α_s can then be written as follows:

$$F_2(x,Q^2) = x\sum_{q,\bar{q}} Q_q^2 \int_x^1 \frac{d\xi}{\xi} q(\xi,\mu_F^2)$$

$$\times\left\{\delta\left(1-\frac{x}{\xi}\right) + \frac{\alpha_s}{2\pi}\left[P^{qq}\left(\frac{x}{\xi}\right)\ln\frac{Q^2}{\mu_F^2} + C_q\left(\frac{x}{\xi}\right)\right]\right\}$$

- The above procedure is called factorization. That factorization works to all orders in perturbation theory is a fundamental property of QCD and content of the factorization theorem.

III. Scaling violation and DGLAP equations

Remarks:

- The necessity of introducing a scale μ_F leads to a violation of Bjorken scaling, i.e. $\partial F_2/\partial Q^2 \neq 0$.
- The distribution $q(x, \mu_F^2)$ can not be calculated, since it contains contributions from the non-perturbative regime. It can however be measured at a given scale (by measuring $F_2, \ldots$).
- The arbitrariness of factorization is reflected in the function C_q. Does this destroy the predictive power of the theory? No! Parton distributions are universal:

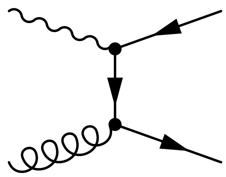
  - Extract the parton distribution functions by comparing experimental data on F_2 obtained in DIS with the theoretical prediction in a fixed factorization scheme.

  - Use these parton distributions to compare, after calculating in the same factorization scheme, theoretical results for cross sections in hadron-hadron collisions with experimental data.

- Moreover, the variation of the parton distributions with the scale can be calculated perturbatively. The equations encoding this variation are called Dokshitzer-Gribov-Lipatov-Altarelli-Parisi equations.

III. Scaling violation and DGLAP equations

To formulate the DGLAP equation for the quark distribution to order α_s, we need to consider also the process $\gamma^* g \to q\bar{q}$:



$+ \ldots$

Result for the gluon structure function:

$$\hat{F}_2^g(x, Q^2) = x \sum_{q\bar{q}} Q_q^2 \frac{\alpha_s}{2\pi} \left(P^{qg}(x) \ln \frac{Q^2}{\kappa^2} + C_g(x) \right)$$

where also a logarithmic singularity is present for vanishing quark virtuality. The gluon ($\to q\bar{q}$) splitting function reads

$$P^{qg}(x) = T_R \left[x^2 + (1-x)^2 \right], \qquad T_R = \frac{1}{2}$$

III. Scaling violation and DGLAP equations

- The singularity is also absorbed into the renormalized quark distribution, and F_2 then reads:

$$F_2(x,Q^2) = x\sum_{q,\bar{q}} Q_q^2 \int_x^1 \frac{d\xi}{\xi} q(\xi,\mu_F^2)$$
$$\times\left\{\delta\left(1-\frac{x}{\xi}\right)+\frac{\alpha_s}{2\pi}\left[P^{qq}\left(\frac{x}{\xi}\right)\ln\frac{Q^2}{\mu_F^2}+C_q\left(\frac{x}{\xi}\right)\right]\right\}$$
$$+x\sum_{q,\bar{q}} Q_q^2 \int_x^1 \frac{d\xi}{\xi} g(\xi,\mu_F^2)$$
$$\times\left\{\frac{\alpha_s}{2\pi}\left[P^{qg}\left(\frac{x}{\xi}\right)\ln\frac{Q^2}{\mu_F^2}+C_g\left(\frac{x}{\xi}\right)\right]\right\}$$

- The DGLAP equation for the quark distribution follows by considering $\mu_F^2 \partial F_2/\partial \mu_F^2 = 0^*$: ($t \equiv \mu_F^2$):

$$t\frac{\partial}{\partial t}q(x,t) = \frac{\alpha_s}{2\pi}\int_x^1 \frac{d\xi}{\xi}\left[P^{qq}\left(\frac{x}{\xi}\right)q(\xi,t)+P^{qg}\left(\frac{x}{\xi}\right)g(\xi,t)\right]$$

---

*Cf. the renormalization group equations that determine the variation of the QCD coupling with the renormalization scale μ_R

III. Scaling violation and DGLAP equations

Including antiquarks, different quark flavours and the gluon gives: (q_i here stands for $u, \bar{u}, d, \bar{d}, \ldots$)

$$t\frac{\partial}{\partial t}\begin{pmatrix} q_i(x,t) \\ g(x,t) \end{pmatrix} = \frac{\alpha_s}{2\pi}\sum_{q_j}\int_x^1 \frac{d\xi}{\xi}$$

$$\times \begin{pmatrix} P_{q_i q_j}\left(\frac{x}{\xi}\right) & P_{q_i g}\left(\frac{x}{\xi}\right) \\ P_{g q_i}\left(\frac{x}{\xi}\right) & P_{gg}\left(\frac{x}{\xi}\right) \end{pmatrix} \begin{pmatrix} q_j(\xi,t) \\ g(\xi,t) \end{pmatrix}$$

At leading order, we have:

$$P_{q_i q_j} = P_{\bar{q}_i \bar{q}_j} = \delta_{ij} C_F \left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x)\right],$$

$$P_{q_i \bar{q}_j} = P_{\bar{q}_i q_j} = 0,$$

$$P_{q_i g} = P_{\bar{q}_i g} = T_R\left[x^2 + (1-x)^2\right],$$

$$P_{g q_i} = P_{g \bar{q}_i} = C_F\left[\frac{1+(1-x)^2}{x}\right],$$

$$P_{gg} = 2C_A\left[\frac{x}{(1-x)_+} + \frac{1-x}{x} + x(1-x)\right] + \delta(1-x)\frac{11C_A - 2n_f}{6}$$

III. Scaling violation and DGLAP equations

The splitting functions or 'evolution kernels' have a very nice probabilistic interpretation:

$P_{ab}(x)$ is the probability to find a parton of type a in a parton of type b with a fraction x of the longitudinal momentum of the parent parton.

Solving the DGLAP equations

A practical method is to choose a reference scale μ_0 and a parametrization, e.g.

$$q(x, \mu_0^2) = Ax^a(1-x)^b\left[1 + \epsilon\sqrt{x} + \gamma x\right]$$

and likewise for $g(x, \mu_0^2)$. Then

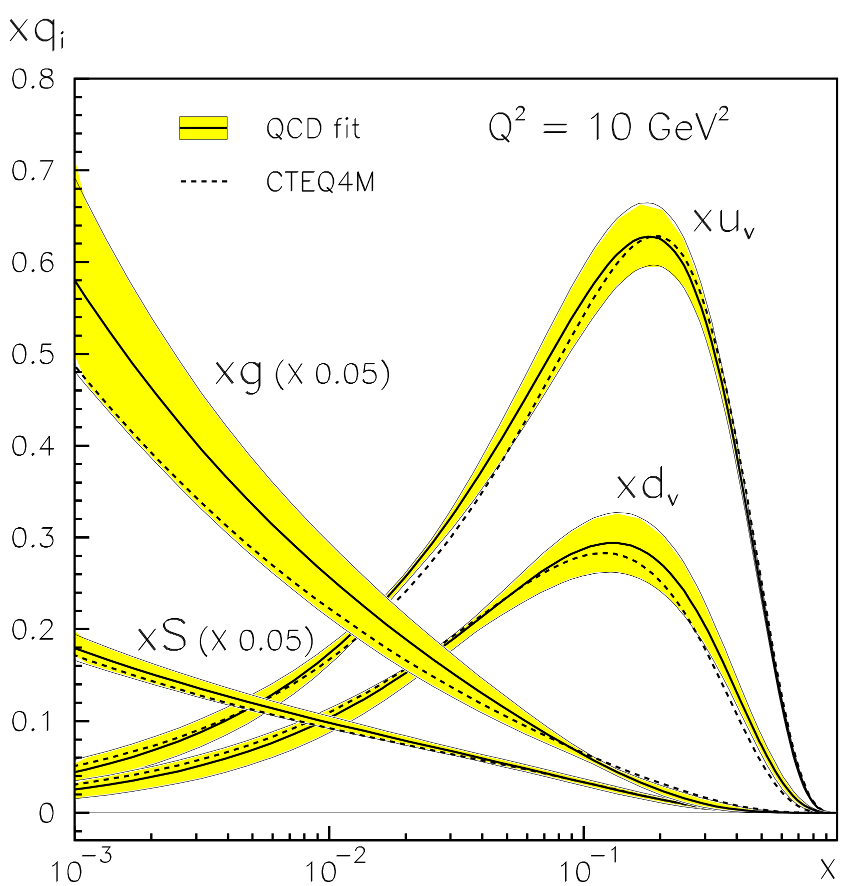
- Solve the integro-differential equation numerically and thus find $q(x, \mu^2)$ at arbitrary scales
- find the optimal values for the parameters $a, b, \ldots$ from a fit to the experimental data
- Problem: Standard fit procedure does not provide PDF uncertainties

Many such parametrizations exist.
See http://www-spires.dur.ac.uk/hepdata/pdf.html

Example for a QCD fit of PDFs:

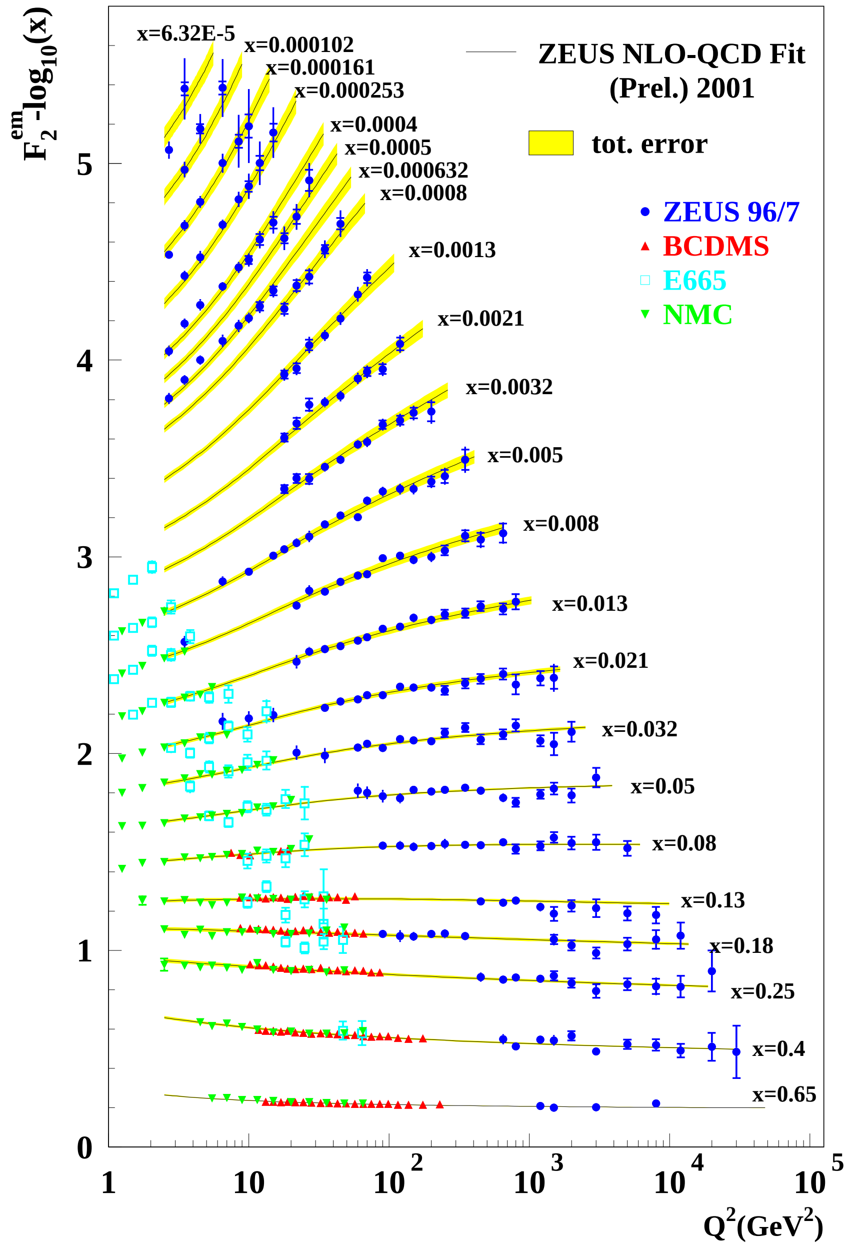
M. Botje, DESY-99-038

- Data: ZEUS, H1; 'old data' from fixed target NC: SLAC, BCDMS, NMC, E665; fixed target CC: CCFR; $pp.pN \to \mu^+\mu^- X$: E866
- fit quality: $\chi^2/\mathrm{d.o.f.} = 1537/1557$
- systematic studies & propagation of experimental uncertainties of input data



III. Scaling violation and DGLAP equations

Scaling violation in $F_2^{\mathrm{e.m.}}$



Qualitative feature of $F_2^{\mathrm{e.m.}}$: At large x, $F_2^{\mathrm{e.m.}}$ decreases with Q^2, at small x, $F_2^{\mathrm{e.m.}}$ increases with Q^2. Intuitively clear: Larger Q^2 $\longrightarrow$ more partons via splitting that have to share overall momentum $\longrightarrow$ more partons with small momenta, less partons with large momenta.

IV. Summary

- Deep inelastic scattering is of relevance for many reactions involving neutrinos
- The structure of nucleons as seen by lepton scattering at high Q^2 can be described by quark and gluon distributions
- Bjorken scaling is naturally described by the parton model
- Precise measurements of the proton structure function have been performed, in particular at HERA
- QCD corrections lead to a violation of Bjorken scaling by terms $\sim \ln(Q^2)$.
- The DGLAP equations, which can be derived from QCD , determine the variation of the parton distribution functions with the (factorization) scale
- Parton distribution functions are obtained by fitting DGLAP-evolved *ansätze* with experimental data. They are universal, i.e. can be used to predict any process involving initial state hadrons